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$3n-5$ EDGES DO FORCE A SUBDIVISION OF K_5

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Dedicated to the memory of Paul Erdős.

We prove the conjecture of G. A. Dirac from 1964, that every graph of order $n \geq 3$ with $3n-5$ edges contains a subdivision of K_5.

1. Introduction and Notation

It has been conjectured by G. A. Dirac in 1964 [2] that every graph on n vertices with at least $3n-5$ edges contains a subdivision of the complete graph K_5 on five vertices. This would be best possible for every $n \geq 5$, as maximal planar graphs show. This conjecture is also mentioned by P. Erdős and A. Hajnal in [3], but it seems that they only report it. In any case, P. Erdős was very interested in this conjecture, and every time we met in the sixties or seventies, he asked me about any progress in it. So I am glad that I could present a solution of this problem on his last conference.

The first result in this context (as far as I know) was obtained by Z. Skupień [10] who proved that Dirac's conjecture is true for locally hamiltonian graphs, i.e. graphs where every vertex has a hamiltonian neighbourhood. Let $|G|$ and $||G||$ denote the number of vertices and edges of a graph G, respectively, and $\dot{G}$ be any subdivision of G. C. Thomassen proved in [11] that every graph G with $||G|| \geq 4|G|-10$ contains a $\dot{K}_5$ what he improved recently in [12] to $||G|| \geq \frac{7}{2}|G|-7$. More exactly, he proved in [12] that every graph G with $||G|| \geq \frac{7}{2}|G|-7$ contains a $\dot{K}_5$ where a prescribed vertex is no branch vertex, and with this condition the result is best possible.

One should mention also that the consequence of our result that every graph G with $||G|| \geq 3|G|-5$ is contractible to K_5, is known for a long time. This

Mathematics Subject Classification (1991): 05C35, 05C10 result follows easily from the nice characterization of all edge-maximal graphs not contractible to K_5, given by K. Wagner in [14].

The aim of our paper is to prove Dirac's conjecture.

Theorem 1. *Every graph G with $||G|| \geq 3|G|-5$ and $|G| \geq 3$ contains a subdivision of K_5.*

I emphasize that in the whole paper the term *graph* always means *finite simple graph*. A. Kézdy and P. McGuinness proved in [4] that a counterexample to Theorem 1 of least order is 5-connected and does not contain a K_4^-, where K_n^- denotes K_n minus one edge. Using this result, Theorem 1 follows at once from

Theorem 2. *Every 5-connected graph G with $||G|| \geq 3|G|-6$ contains a K_4^- or a $\dot{K}_5$.*

We will prove this theorem in sections 3 and 4. The main tool in the proof will be the result from [6], that every graph G of girth $\tau(G) \geq 5$ with $||G|| \geq 2|G|-5$ and $|G| \geq 6$ contains a $\dot{K}_5^-$ or is the Petersen graph. We will consider a counterexample G to Theorem 2 and a maximal connected subgraph H_0 of G so that for the contraction $G_0 := G/H_0$ still $||G_0|| \geq 3|G_0|-6$ and $|G_0| \geq 5$ hold. Let H_1 be the graph spanned by the neighbours of H_0 in G. This graph H_1 does not contain a $\dot{K}_5^-$, since such a $\dot{K}_5^-$ could be extended to a $\dot{K}_5 \subseteq G$, using an appropriate path through H_0. We will prove some properties of G_0 and H_1 in section 3, in particular, $\tau(H_1) \geq 5$. In section 4, we will apply the above mentioned result from [6] to a certain subgraph $\overline{F}_0$ of H_1. Section 2 contains known results necessary for the proof.

Since Theorem 2 refers to graphs G with $||G|| \geq 3|G|-6$, one can use it also for a characterization of all graphs G with $||G|| = 3|G|-6$ not containing a $\dot{K}_5$. All these graphs are obtained by pasting disjoint maximal planar graphs together along triangles. We postpone this determination of the extremal graphs to a later paper.

We need some further notation. For a positive integer $n \in \mathbb{N}$ and a set M, we define $\mathbb{N}_n := \{m \in \mathbb{N} : m \leq n\}, \mathfrak{P}_n(M) := \{A \subseteq M : |A| = n\}$, and $\mathfrak{P}_{\leq n}(M) := \{A \subseteq M : |A| \leq n\}$. As mentioned above, a *graph* has neither multiple edges nor loops. So we can consider an edge an element of $\mathfrak{P}_2(V(G))$, but we write $[x,y]$ for the edge connecting x and y. Correspondingly, for $A, B \subseteq V(G), [A,B]_G := \{[a,b] \in E(G) : a \in A \text{ and } b \in B\}$. For $A \subseteq V(G)$ or a subgraph $A \subseteq G$, let $G(A)$ be the subgraph of G spanned by $A, G-A := G(V(G)-V(A))$, and $N_G(A) := \{x \in G-A :$ there is an $[a,x] \in E(G)$ with $a \in A\}$, where $x \in G$ for a graph G always means $x \in V(G)$. For $A = \{a\}$, we simply write $G-a, N_G(a)$ etc. and we delete a subscript G, if it seems obvious which graph is meant. For subgraphs $A_1, A_2 \subseteq G$, *disjoint* means vertex disjoint and is denoted by $A_1 \cap A_2 = \emptyset$. Let $\delta(G) := \min_{x \in G} d_G(x)$ denote the minimum degree of G and define $V_n(G) := \{x \in G : d_G(x) = n\}$ and $V_{\geq n}(G) := \{x \in G : d_G(x) \geq n\}$ for $n \in \mathbb{N}$. The set of components of G is denoted by $\mathfrak{C}(G)$ and a $C \in \mathfrak{C}(G)$ is called *trivial*, if $|C| = 1$. Set $c(G) := |\mathfrak{C}(G)|$. For a graph G and $E' \subseteq \mathfrak{P}_2(V(G)), G \cup E'$ is the *graph* $(V(G), E(G) \cup E')$, i.e. also an

$[x,y] \in E(G) \cap E'$ remains a simple edge. For $H \subseteq G$, the *graph* G/H arises from G by identifying $V(H)$ to a $z \in H$ (deleting all arising loops and multiple edges), hence $V(G/H) = V(G-H) \cup \{z\}$; for connected H, we call it a *z-contraction* and say, H is contracted to z. We call a contraction G/H *proper*, if $|H| \geq 2$, and for $[x,y] \in E(G), G/_{[x,y]} := G/H$ for $H := (\{x,y\},\{[x,y]\})$. For a graph G with $\delta(G) \geq 3, B(\dot{G}) := \{x \in \dot{G} : d_{\dot{G}}(x) \geq 3\}$ denotes the set of branch vertices of the subdivision $\dot{G}$ of G.

A path and a circuit pass through every vertex at most once. A path $P: x_0, x_1, \dots, x_n$ is given by the sequence of its consecutive vertices $x_0, x_1, \dots, x_n$, but is considered as a subgraph. If the path P has endvertices a und b, we call P an *a,b-path* and define $\overset{\circ}{P} := P - \{a,b\}$; we have always $a \neq b$ or $|P| = 1$. For vertices x,y of a path $P, P[x,y]$ denotes the x,y-path contained in P, and similarly, $P[x,y) := P[x,y] - y$. For $A_i \subseteq V(G)$ or $A_i \subseteq G$ for $i = 1,2$, an *A_1, A_2-path* is an a_1, a_2-path P with $V(P) \cap V(A_i) = \{a_i\}$ for $i = 1,2$ and an *A_1-path* is an a_1, a_2-path P with $a_1 \neq a_2$ and $V(P) \cap V(A_1) = \{a_1, a_2\}$. For $A_1 = \{a_1\}$, say, we write simply a_1, A_2-path, and if $a_1 \in A_2$, the only a_1, A_2-path is the path consisting of a_1. For distinct vertices $x,y \in G, \kappa(x,y;G)$ denotes the maximum number of openly (or internally) disjoint x,y-paths in G, and $\kappa(G) := \min_{x \neq y} \kappa(x,y;G)$ is the connectivity number of $G \ncong K_1$ and $\kappa(K_1) = 0$. So every n-connected graph G has $|G| \geq n+1$. For $x \in G$ and $A \subseteq V(G-x)$ or a graph A with $V(A) \subseteq V(G-x)$, an *x,A-fan of order n* in G is a set of $x, V(A)$-paths $P_1, \dots, P_n$ in G with $V(P_i) \cap V(P_j) = \{x\}$ for all $\{i,j\} \in \mathfrak{P}_2(\mathbb{N}_n)$. We call $a_1, \dots, a_n$ the *endvertices of the x,A-fan* $P_1, \dots, P_n$, if $V(P_i) \cap V(A) = \{a_i\}$ for $i \in \mathbb{N}_n$, and say, the fan ends in $\{a_1, \dots, a_n\}$ and P_i ends in a_i. Sometimes it is convenient to admit also $x \in A$. If $x \in A$, then an x,A-fan of order n consists of an $x,(A - \{x\})$-fan of order $n-1$ and the x,x-path.

A set $T \subseteq V(G)$ *separates the graph G minimally*, if $G - T$ is disconnected, but $G - T'$ is connected for all $T' \subsetneqq T$. *A* set $T \subseteq V(G)$ *separates G trivially*, if $c(G-T) = 2$ and $G - T$ has a trivial component. If $x \in G$ and $A \subseteq V(G-x)$, we say, a $T \subseteq V(G)$ *separates x from A*, if $x \notin T$ and $C \cap A = \emptyset$ for the $C \in \mathfrak{C}(G-T)$ containing x. F is a *T-fragment of G*, if there is a non-empty $\mathfrak{C}' \subsetneqq \mathfrak{C}(G-T)$ such that $F = \bigcup_{C \in \mathfrak{C}'} C$ holds. In particular, if F is a T-fragment, then T separates G and $\widetilde{F} := \bigcup_{C \in \mathfrak{C}''} C$ for $\mathfrak{C}'' := \mathfrak{C}(G-T) - \mathfrak{C}'$ is also a T-fragment, the *complementary T-fragment*. Furthermore, we define, $\overline{F} := G(F \cup T)$ for a T-fragment F of G. Since $\overline{F}$ depends on G and T, we will normally define ad hoc, what is meant by $\overline{F}$. If T is minimally separating G and F is a T-fragment of G, then $N_G(F) = T$.

2. Presumed results

In this section we put together the results we need in the proof of Theorem 2. First of all some results from [6].

Theorem M [6]**.** *Every graph G with $\tau(G)\geq 5, ||G||\geq 2|G|-5$, and $|G|\geq 5$, which is neither the pentagon nor the Petersen graph, contains a $\dot{K}_5^-$.*

We will use this result also in the following form.

Corollary M. *Let G be a graph with $||G|| \geq 3|G|-8$ and $|G|\geq 6$ such that there is a $z\in G$ with $N_G(z)=V(G-z)$ and $\tau(G-z)\geq 5$. Then there is a $\dot{K}_5\subseteq G$ with $z\notin B(\dot{K}_5)$ or $G-z$ is the pentagon or the Petersen graph.*

Proof. Since $||G-z||=||G||-|G-z|\geq 3|G-z|-5-|G-z|=2|G-z|-5$, we can apply Theorem M to $G-z$ and get a $\dot{K}_5^-\subseteq G-z$, if $G-z$ is neither the pentagon nor the Petersen graph. Joining the vertices a,b of degree 3 in $\dot{K}_5^-\subseteq G-z$ by the path a,z,b, we get $\dot{K}_5\subseteq G$ as wanted. ∎

We need further the following property of the Petersen graph.

Lemma Pet [6]**.** *In the Petersen graph P, for every $A\in\mathfrak{P}_3(V(P))$, there is a $\dot{K}_4\subseteq P$ with $A\subseteq B(\dot{K}_4)$.*

We need a concept of connectedness stronger than n-connectivity, but weaker than $(n+1)$-connectivity. We call a graph G *n^+-connected* for an $n\in\mathbb{N}$, if $\kappa(G)\geq n$ and every $S\in\mathfrak{P}_n(V(G))$ separating G separates G in a trivial way.

Lemma M [6]**.** *Let G be an n^+-connected graph for some $n\in\mathbb{N}$.*

(a) *Assume $x\in G$ with $d(x)\geq n+1$ and $A\subseteq V(G-x)$ with $|A|\geq n+1$. Then there is an x,A-fan of order $n+1$ in G or $|A|=n+1$ and there is an $a\in A$ with $N_G(a)=A-\{a\}$.*

(b) *If $|G|\geq n+4$, for all distinct $x_1,x_2\in G$ with $d(x_i)\geq n+1$ for $i=1$, 2, $\kappa(x_1,x_2;G)\geq n+1$ holds.*

The following consequence of Menger's theorem is due to H. Perfect [9].

Theorem P [9]**.** *Let x be a vertex of a graph G and $A\subseteq V(G-x)$. If there are an x,A-fan of order k ending in $a_1,\dots,a_k$ and an x,A-fan of order $n\geq k$ in G, then there is also an x,A-fan in G of order n ending in $A'\supseteq\{a_1,\dots,a_k\}$ (with $|A'|=n$).*

At the end of the proof we will apply the following result due to M. E. Watkins and D. M. Mesner [15].

Theorem WM [15]**.** *Let G be a 2-connected graph and assume $A=\{a_1,a_2,a_3\}\in\mathfrak{P}_3(V(G))$ is not contained in a circuit of G. Then at least one of the following three cases occurs.*

(i) *There are an* $S=\{s^1,s^2\}\in\mathfrak{P}_2(V(G)-A)$ *and a* $C_i\in\mathfrak{C}(G-S)$ *with* $V(C_i)\cap A=\{a_i\}$ *for* $i\in\mathbb{N}_3$*;*

(ii) *There is an* $S=\{s_1^1,s_2^1,s_3^1,s^2\}\in\mathfrak{P}_4(V(G)-A)$ *such that there is a* $C_i\in\mathfrak{C}(G-\{s_i^1,s^2\})$ *with* $V(C_i)\cap(A\cup S)=\{a_i\}$ *for* $i\in\mathbb{N}_3$*;*

(iii) *There are disjoint* $S^1=\{s_1^1,s_2^1,s_3^1\}$ *and* $S^2=\{s_1^2,s_2^2,s_3^2\}$ *in* $\mathfrak{P}_3(V(G)-A)$ *so that there are* $\{s_i^1,s_i^2\}$*-fragments* C_i *of* G *with* $V(C_i)\cap(A\cup S^1\cup S^2)=\{a_i\}$ *such that* $\mathfrak{C}((G-\bigcup_{i=1}^{3}V(C_i))-\{[s_i^1,s_i^2]:i\in\mathbb{N}_3\})=\{C^1,C^2\}$ *with* $S^i\subseteq V(C^i)$ *for* $i=1,2$.

Note that also in case (ii) and (iii), $C_i\cap C_j=\emptyset$ for $i\neq j$, since $\kappa(G)\geq 2$ and for $i\in\mathbb{N}_3, N(C_i)=\{s_i^1,s^2\}$ and $C_i\cap S=\emptyset$ in case (ii) and $N(C_i)=\{s_i^1,s_i^2\}$ and $C_i\cap(S^1\cup S^2)=\emptyset$ in case (iii).

Remark. Watkins and Mesner dit not state their result in [15] for specified vertices a_1,a_2,a_3, but their proof works for this specification (cf. "Concluding remarks" in [15]). Theorem WM is also an easy consequence of the "H-paths Theorem" in [5], since a system of three openly disjoint A-paths in G, in every $a\in A$ ending exactly 2, is equivalent to a circuit containing A. In the notation of [5], then the cases (i), (ii), and (iii) correspond to $|C_0|=2,1$, and 0 respectively.

When dealing with Theorem WM, it is convenient to use a common notation for the different cases. So we denote always $N(C_i)=:\{s_i^1,s_i^2\}=:S_i$ for $i\in\mathbb{N}_3$. Hence in the cases (i) and (ii) we have $s_1^1=s_2^1=s_3^1=:s^1$ or $s_1^2=s_2^2=s_3^2=:s^2$. Similarly, we write $S^i:=\{s_j^i:j\in\mathbb{N}_3\}$ for $i=1,2$ also in (i) and (ii) and $C^i:=(S^i,\emptyset)$, if $|S^i|=1$. In case (ii), $G-s^2$ is connected, hence also $C^1:=(G-s^2)-\bigcup_{i=1}^{3}V(C_i)$ is connected. But for symmetry in the arguments, in general, we do not assume $|S^2|=1$ in case (ii), but only $|S^1|=1$ or $|S^2|=1$. In the cases (i) and (ii), the C_i are defined as components, but it is more convenient to enlarge them to fragments as in case (iii). We will assume always that in case (i) C_1,C_2,C_3 are S-fragments of G forming a partition of $G-S$ and that in case (ii) C_j is the union of *all* $C\in\mathfrak{C}(G-\{s_j^1,s^2\})$ with $V(C)\cap(A\cup S)\subseteq\{a_j\}$, as we have it per definition in (iii). So in all cases, C_1,C_2,C_3,C^1,C^2 form a partition of $V(G)$, where C^1 in case (ii) is defined as above, but using the *fragments* C_i. Furthermore, $\overline{C}_i:=G(C_i\cup S_i)$ for $i\in\mathbb{N}_3$, as defined in general.

Before we introduce further notation, we will show that C^1 and C^2 have nicer properties, if we minimize $|C^1|+|C^2|$.

Corollary WM. *If we choose in Theorem* WM *the case and the partition* C_1,C_2,C_3,C^1,C^2 *so that* $|C^1|+|C^2|$ *is least possible, then*

(ii) S^i *is contained in a block of* C^i *for* $i=1,2$ *and even*

(iii) $\kappa(C^i)\geq 2$ *for* $i=1,2$, *if* $|C^1|>1$ *and* $|C^2|>1$.

Proof. We may assume $|C^1|>1$, hence $|S^1|=3$. As we have seen above, C^1 is connected. Suppose there is a $s\in C^1$ separating C^1. If for all $C\in\mathfrak{C}(C^1-s)$, $|C\cap S^1|\leq 1$ holds, then we can take in Theorem WM the vertex s instead of S^1 and come to a corresponding partition D_1,D_2,D_3,D^1,C^2 with $|D^1|<|C^1|$ (and from case (ii) to case (i) or from case (iii) to case (ii)). This contradiction shows that there is a $C_0\in\mathfrak{C}(C^1-s)$ with $|C_0\cap S^1|\geq 2$. If there is a $C\neq C_0$ in $\mathfrak{C}(C^1-s)$ with $C\cap S^1\neq\emptyset$, say, $s_1^1\in C$, then we can take in Theorem WM the vertex s instead of s_1^1 and come to a corresponding partition D_1,D_2,D_3,D^1,C^2 with $|D^1|<|C^1|$ (but remain in the same case). This contradiction shows $S^1\subseteq V(C_0)\cup\{s\}$. Since this holds for every separating $s\in C^j$ and $j\in\mathbb{N}_2$ (with $|C^j|>1$), (ii) follows. Continuing the above consideration, there is a $C\neq C_0$ in $\mathfrak{C}(C^1-s)$, hence $C\cap S^1=\emptyset$. But for such a C we must have $N_G(C)\cap C^2\neq\emptyset$, since $\kappa(G)\geq 2$ and $N_G(\bigcup_{i=1}^{3} C_i)\cap V(C^1)=S^1$. But this cannot happen in case (iii) of Theorem WM, which proves (iii). ∎

Applying Theorem WM, we will choose C_1,C_2,C_3,C^1,C^2 as in Corollary WM and use the following notation. If $|C^i|>1$, then by (ii) of Corollary MW there is an $s^i_j,(S^i-\{s^i_j\})$-fan P^i_{jk},P^i_{jl} in C^i ending in s^i_k,s^i_l for all $j\in\mathbb{N}_3$, where we have set $\mathbb{N}_3=\{j,k,l\}$. If $|C^i|=1$, we use this notation P^i_{jk} for the path of length 0 in C^i. Since $\kappa(G)\geq 2$, we have in all cases for every $j\in\mathbb{N}_3$ an a_j,S_j-fan P^1_j,P^2_j in $\overline{C}_j$ ending in s^1_j,s^2_j.

3. Some properties of a counterexample to Theorem 2

In this section we start the proof of Theorem 2. We consider a "minimal contraction" of a counterexample to Theorem 2 and deduce some of its most important properties.

We assume that there is a 5*-connected graph* G *with* $||G||\geq 3|G|-6$ *containing neither a* K_4^- *nor a* $\dot{K}_5$. Consider $\mathcal{H}:=\{H\subseteq G : H$ connected with $||G/H||\geq 3|G/H|-6$ and $|G/H|\geq 5\}\neq\emptyset$. We choose H_0 maximal in $(\mathcal{H},\subseteq)$. Then H_0 is an induced subgraph of G. Denote $G_0:=G/H_0$ and let z_0 be the vertex of $V(G_0)-V(G)$. Define $N_1:=N_{G_0}(z_0)=N_G(H_0)$ and $H_1:=G_0(N_1)=G(N_1)$. We adopt the following notation: for every $F\subseteq G_0$ or $F\subseteq V(G_0)$, $'F:=F-\{z_0\}$. We often use the following property of G_0.

(3.1) *If* $F\subseteq G_0$ *is connected with* $z_0\in F$ *and* $2\leq|F|\leq|G_0|-4$, *then* $||G_0/F||\leq 3|G_0/F|-7$ *holds.*

Proof. $H := G(H_0 \cup {}'F)$ is also connected and $G/H \cong G_0/F$. Since $|G_0/F| \geq 5$ and $H \supsetneqq H_0$, the choice of H_0 implies $||G/H|| < 3|G/H| - 6$. ∎

Since $|G_0| = 5$ would imply $K_4^- \subseteq {}'G_0 \subseteq G$, we have $|G_0| \geq 6$. Therefore, contracting an edge $[z_0, x] \in E(G_0)$, by (3.1) we loose more than three edges. So we have shown

(3.2) *$\delta(H_1) \geq 3$.*

Since $\kappa(G) \geq 5$, (3.2) and $|G_0| \geq 6$ imply

(3.3) *$\delta(G_0) \geq 4$ and $d_{G_0}(x) \geq 5$ for all $x \in G_0 - N_1$.*

Next we prove $\kappa(G_0) \geq 4$ and something more.

(3.4) (a) $\kappa(G_0) \geq 4$

(b) *If $T \in \mathfrak{P}_4(V(G_0))$ separates G_0, then $z_0 \in T$, $||G_0({}'T)|| \leq 1$, and $d_{G_0(T)}(z_0) \geq 2$, and in the case $d_{G_0(T)}(z_0) = 2$, ${}'T$ is independent.*

Proof. Assume that T is a minimally separating vertex set of G_0 with $|T| \leq 4$. Since $\kappa(G) \geq 5, T$ does not separate G, hence $z_0 \in T$. Set $d_0 := d_{G_0(T)}(z_0)$. For any $C \in \mathfrak{C}(G_0 - T)$, define $G_1 := G_0(C \cup T) = \overline{C}$ and $G_2 := G_0 - V(C) = \overline{\overline{C}}$. Then $|G_i| \geq 5$ for $i = 1, 2$ by (3.3). Since T is minimally separating, $F_i := G_i - {}'T$ is connected for $i = 1, 2$. Let G_i' arise from G_0 by contracting F_i to z_0. Then (3.1) implies

$$||G_i'|| \leq 3|G_i'| - 7 \tag{3.4.1}$$

for $i = 1, 2$. Since $[z_0, t] \in E(G_i')$ for all $t \in {}'T$, addition of (3.4.1) for $i = 1$ and $i = 2$ gives

$$||G_0|| + d_0 + 2(|{}'T| - d_0) + ||G_0({}'T)|| \leq 3(|G_0| + |T|) - 14. \tag{3.4.2}$$

Using $||G_0|| \geq 3|G_0| - 6$, (3.4.2) implies

$$|T| + d_0 \geq 6 + ||G_0({}'T)||. \tag{3.4.3}$$

Since $d_0 < |T|$, this implies $|T| = 4$, and then $d_0 \geq 2, ||G_0({}'T)|| \leq 1$ and $||G_0({}'T)|| = 0$ for $d_0 = 2$. ∎

We need similar results for certain z_0-contractions of G_0.

(3.5) *Let $H \subseteq G_0$ be connected with $z_0 \in H$ and $2 \leq |H| \leq |G_0| - 5$ and let G_0' arise from G_0 by contraction of H to z_0. Assume $||G_0'|| \geq 3|G_0'| - 7$. Then the following statements hold.*

(a) *$\delta(G_0'(N_{G_0'}(z_0))) \geq 2$;*

(b) $\kappa(G'_0)\geq 3$ *or* $|G'_0|=6, d_{G'_0}(z_0)=5$, *and* G'_0-z_0 *consists of two triangles sharing exactly one vertex;*

(c) *If* $T\in\mathfrak{P}_3(V(G'_0))$ *minimally separates* G'_0, *then* $z_0\in T$, ${}'T\subseteq N_{G'_0}(z_0)$, *and if* $||G({}'T)||=1$, *then* T *trivially separates* G'_0.

Proof. By (3.1) and assumption we have $||G'_0||=3|G'_0|-7$. So we loose at least 3 edges contracting an edge $[z_0,x]\in E(G'_0)$, since we can apply (3.1) to the connected $F:=G_0(H\cup x)$ for $x\in N_{G'_0}(z_0)\neq\emptyset$ and $G_0/F\cong G'_0/_{[z_0,x]}$ holds. This proves (a).

Suppose now that $T\subseteq V(G'_0)$ with $|T|\leq 3$ minimally separates G'_0. Since T does not separate G_0 by (3.4)(a), $z_0\in T$. For any T-fragment F of G'_0, we define the *graphs* $G_1:=\overline{F}\cup\{[z_0,t]:t\in{}'T\}$ and $G_2:=\overline{\overline{F}}\cup\{[z_0,t]:t\in{}'T\}$. As T separates G'_0 minimally, G_i is a z_0-contraction of G'_0, hence a proper z_0-contraction of G_0 for $i=1,2$. So (3.1) implies

$$||G_i||\leq 3|G_i|-7 \quad\text{or}\quad G_i\cong K_4 \quad\text{for}\quad i\in\mathbb{N}_2, \tag{3.5.1}$$

since $|G_i|\geq 4$ for $i\in\mathbb{N}_2$ by (3.3) and (a), and $||G_i||>3|G_i|-7$ implies $G_i\cong K_4$ for $|G_i|=4$.

Set $\delta_i:=3|G_i|-||G_i||$ for $i=1,2$ and $d_0:=d_{G'_0(T)}(z_0)$. Then $\delta_i\geq 7$ or $\delta_i=6$ and $G_i\cong K_4$ by (3.5.1). By addition we get

$$||G'_0||+||G'_0(T)||+2(|{}'T|-d_0)=||G_1||+||G_2||=3|G'_0|+3|T|-\delta_1-\delta_2 \tag{3.5.2}$$

Using $||G'_0||=3|G'_0|-7$, (3.5.2) implies

$$\delta_1+\delta_2+||G'_0(T)||+2(|{}'T|-d_0)=3|T|+7. \tag{3.5.3}$$

Let us first assume $|T|\leq 2$. Then $|T|=2$ by (3.5.3) and for an $i\in\mathbb{N}_2,\delta_i=6$ and $G_i\cong K_4$, say, for $i=1$. Since $||G'_0(T)||+2(|{}'T|-d_0)\geq 1$, also $\delta_2=6$ follows and $G_2\cong K_4$ by (3.5.1). But then $||G'_0(T)||+2(|{}'T|-d_0)=1$, i.e. $d_0=1$, and (b) follows.

Now assume $|T|=3$. $\delta_1=\delta_2=6$ would imply $K_4^-\subseteq G$ by (3.5.1). So $\delta_i=6$ for at most one $i\in\mathbb{N}_2$. Hence $3\geq ||G'_0(T)||+2(|{}'T|-d_0)\geq 2$. If $d_0<2, ||G'_0(T)||+2(|{}'T|-d_0)=3$ and $||G'_0({}'T)||=0$, hence $d_{i_0}=6$ for an $i_0\in\mathbb{N}_2$ and so $G_{i_0}\cong K_4$ by (3.5.1), which implies $||G'_0({}'T)||=1$, a contradiction. Therefore, $d_0=2$ holds. If $||G'_0({}'T)||=1$, then $||G'_0(T)||=3$, and there is an $i\in\mathbb{N}_2$ with $\delta_i=6$, hence $G_i\cong K_4$ by (3.5.1), say, $i=1$. Since $K_4^-\not\subseteq G, |C|\geq 2$ for all components $C\neq F$ of G'_0-T. This means $c(G'_0-T)=2$, since we have seen that there is no T-fragment L of G'_0 with $|L|\geq 2$ and $|\widetilde{L}|\geq 2$. Hence T separates G'_0 trivially. ∎

Our next goal worth mentioning is to show $\tau(H_1)\geq 5$. For this we need some further notation. For $\{x,y\}\in\mathfrak{P}_2(N_1)$, there is an x,y-path in $G(H_0\cup\{x,y\})-[x,y]$, since H_0 is connected and $\{x,y\} \subseteq N_G(H_0)$. We denote such a path by xH_0y and call it an x,y-path through H_0. For $X \in \mathfrak{P}_3(N_1)$, we define $H_0(X) := \{z \in H_0$: there is an z, N_1-fan of order 3 in G ending in $X\}$. Since H_0 is connected and $X \subseteq N(H_0), H_0(X) \neq \emptyset$ for all $X \in \mathfrak{P}_3(N_1)$. For $X = \{x_1,x_2,x_3\}$, we write $H_0(x_1,x_2,x_3)$ instead of $H_0(X)$. Suppose $z \in H_0(x_1,x_2,x_3)$ and set $\{x_1,x_2,x_3\} =: X$. Since $\kappa(G) \geq 5$, by Theorem P, there is a z, N_1-fan $P_1,\ldots,P_5$ of order 5 in G ending in x_1,x_2,x_3,x_4,x_5. Of course, $P_i - x_i \subseteq H_0$ for all $i \in \mathbb{N}_5$. We denote such a fan by $\mathfrak{F}^{x_4,x_5}(H_0(X:z))$ and write also $H_0(x_1,x_2,x_3:z)$ instead of $H_0(X:z)$. A z, N_1-fan of order 4 in G ending in x_1,x_2,x_3,x_4 is denoted by $\mathfrak{F}^{x_4}(H_0(X:z))$. On the other side, we write $H_0(X:z)$ also for the following subset of $N_1 \cup \mathfrak{P}_2(N_1)$: $\{x \in N_1$: there is an $\mathfrak{F}^{x}(H_0(X:z))$ in $G\} \cup \{\{x,y\} \in \mathfrak{P}_2(N_1)$: there is an $\mathfrak{F}^{x,y}(H_0(X:z))$ in $G\}$. Of course, $\{x_4,x_5\} \in H_0(X:z)$ implies $x_4,x_5 \in H_0(X:z)$, but, in general, not vice versa. But we have

(3.6) *If $x \in H_0(X:z)$ for an $X \in \mathfrak{P}_3(N_1)$ and $z \in H_0(X)$, then there is a $y \in N_1 - (X \cup \{x\})$ such that $\{x,y\} \in H_0(X:z)$ holds.*

Proof. By definition of $x \in H_0(X:z)$, there is a z, N_1-fan of order 4 in G ending in $X \cup \{x\}$. Since $\kappa(G) \geq 5$, by Theorem P, there is a z, N_1-fan of order 5 in G ending in $X \cup \{x\}$ and one further vertex $y \in N_1$. Then $\{x,y\} \in H_0(X:z)$ holds. ∎

It is easy to prove now

(3.7) $\tau(H_1) \geq 4$.

Proof. Suppose $X \in \mathfrak{P}_3(N_1)$ spans a triangle in H_1. There are a $z \in H_0(X)$ and a $y \in H_0(X:z)$. By (3.4)(a), there is a y, X-fan $\mathfrak{F}_3$ of order 3 in $G_0 - z_0 = G - V(H_0)$. Then the triangle $G(X)$ and the fans $\mathfrak{F}_3$ and $\mathfrak{F}^{y}(H_0(X:z))$ form a $\dot{K}_5 \subseteq G$ with $B(\dot{K}_5) = X \cup \{y,z\}$, contradicting our assumption on G. ∎

We get as a Corollary

(3.8) *Assume $T \in \mathfrak{P}_4(V(G_0))$ separates G_0.*

(a) *Then $|C| \neq 2$ for all $C \in \mathfrak{C}(G_0 - T)$.*

(b) *If there is a trivial $C \in \mathfrak{C}(G_0 - T)$, then $'T$ is independent.*

Proof. Be $C \in \mathfrak{C}(G_0 - T)$ with $|C| \leq 2$. Since $z_0 \in T$ by (3.4)(b), we conclude $C \subseteq H_1$ from (3.3). If $|C| = 1, N_{H_1}(C) = {'T}$ by (3.2) and $'T$ independent by (3.7). $|C| = 2$ is impossible, since for $[x,y] \in E(C) \subseteq E(H_1)$ we have $|N_{H_1}(\{x,y\})| \geq 4$ by (3.2) and (3.7). ∎

For getting a $\dot{K}_5 \subseteq G$, we start from subdivisions of certain graphs in fragments of G_0 with branch vertices in H_1. First, a notation. For distinct elements a_1,a_2,a_3,a_4, define

$$D(a_1,a_2,a_3,a_4) := (\{a_1,a_2,a_3,a_4\}, \{[a_1,a_2],[a_2,a_3],[a_3,a_1],[a_3,a_4]\}).$$

So $\dot{D}(a_1,a_2,a_3,a_4)$ consists of a circuit through a_1,a_2,a_3 and an a_3,a_4-path. A typical argument in the proof is as follows. Consider any separating set $T = \{a_1,a_2,a_4,z_0\}$ of G_0, a $z \in H_0(a_1,a_2,a_4)$, and an $a_3 \in H_0(a_1,a_2,a_4 : z)$. Let $C \in \mathfrak{C}(G_0 - T)$ with $a_3 \in C$. If we can find $\dot{D}(a_1,a_2,a_3,a_4) \subseteq '\overline{C}$ and an $a_4,\{a_1,a_2\}$-fan of order 2 in $'\overline{\overline{C}}$, then these configurations form a $\dot{K}_4 \subseteq 'G_0$ with $B(\dot{K}_4) = \{a_1,a_2,a_3,a_4\}$. Then we could attach an $\mathfrak{F}^{a_3}(H_0(a_1,a_2,a_4 : z))$ and would get a $\dot{K}_5 \subseteq G$ with $B(\dot{K}_5) = B(\dot{K}_4) \cup \{z\}$, contradicting our assumption on G. In most cases, this last step of addition of a z,N_1-fan of order 4 in G ending in the branch vertices of a $\dot{K}_4 \subseteq 'G_0$ for a $z \in H_0$ will be obvious, so that we will not mention it.

G_0 is 4-connected by (3.4)(a), but this is not enough for the proof. I could not show $\kappa(G_0) \geq 5$, but it is possible to prove that G_0 is 4^+-connected. Originally I used this, and the proof of this fact was complicated and as long as the whole paper is now. But it turned out that it suffices to work in a T-fragment of G_0 which becomes 4^+-connected by "completing" T. We will do this in the next section. First we study more in detail the separating sets with 4 vertices and define $\mathfrak{T}_0 := \{T \in \mathfrak{P}_4(V(G_0)) : T \text{ separates } G_0\}$.

(3.9) *If there are a $T \in \mathfrak{T}_0$ and a $y \in N_1$ with $|N_{H_1}(y) \cap T| \geq 2$, then $c(G_0 - T) = 2$.*

Proof. Suppose there are $T \in \mathfrak{T}_0$ and $y \in N_1$ as above, but $n := c(G_0 - T) \geq 3$ components $C_1,\dots,C_n$ of $G_0 - T$ exist. By (3.4)(b), $y \notin T$, say $y \in C_1$. Set $'T = \{t_1,t_2,t_3\}$ with $\{t_1,t_2\} \subseteq N_{H_1}(y)$. There are a $z \in H_0(y,t_1,t_2)$ and an $\{x,x'\} \in H_0(y,t_1,t_2 : z)$, say, $x \neq t_3$. First assume $x \in C_i$ for an $i \geq 2$. By (3.4)(a), there is an $x,\{t_1,t_2,y\}$- fan $\mathfrak{F}_3$ of order 3 in $'\overline{C_i} \cup ' \overline{C_1}$. Using $\mathfrak{F}_3$ and a t_1,t_2-path through $'\overline{C}_j - t_3$ for a $j \in \mathbb{N}_n - \{1,i\} \neq \emptyset$, we get a $\dot{K}_4 \subseteq 'G_0$ with $B(\dot{K}_4) = \{y,t_1,t_2,x\}$, hence a $\dot{K}_5 \subseteq G$. Therefore, $x \in C_1$. Since there is an x,y-path in C_1, by (3.4)(a) and Theorem P, we find an $x,'T \cup \{y\}$ -fan of order 3 in $'\overline{C}_1$ ending in y,t_{i_1},t_{i_2}. Since $n \geq 3$, it is easy to find now a $\dot{K}_4 \subseteq 'G_0$ with $B(\dot{K}_4) = \{y,t_1,t_2,x\}$, hence a $\dot{K}_5 \subseteq G$. ∎

We need a further notation. If F is a T-fragment of G_0 for any $T \in \mathfrak{T}_0$, then the *graph* F^* is defined by $F^* := \overline{F} \cup \{[z_0,t] : t \in 'T\}$. F^* is the graph we get from G_0 by contracting the connected graph $G_0(\widetilde{F} \cup z_0)$ to z_0. So by (3.1) we have

(3.10) $||F^*|| \leq 3|F^*| - 7$ *for every T-fragment F of G_0 for $T \in \mathfrak{T}_0$.*

(3.11) *Let F_1,F_2 be complementary T-fragments of G_0 for a $T \in \mathfrak{T}_0$.*

(a) *Then $||F_i^*|| \geq 3|F_i^*| - 8$ for $i = 1,2$ and there is an $i \in \mathbb{N}_2$ with $||F_i^*|| = 3|F_i^*| - 7$. If $'T$ is not independent or $'T \nsubseteq N_1$, then $||F_i^*|| = 3|F_i^*| - 7$ for $i = 1$ and $i = 2$.*

(b) *If $||F_1^*|| = 3|F_1^*| - 7$ holds, then $|F_1| \geq 3, |N_{\overline{F}_1}(t) \cap N_1| \geq 2$ for all $t \in 'T$, and $\kappa(F_1^*) \geq 3$.*

(c) *Assume* $||F_1^*||=3|F_1^*|-8$ *and* $|F_1|\geq 2$. *Then* $d_{F_1^*}(t)\geq 3$ *for at least two* $t\in {'T}$. *If* $d_{F_1^*}(t)\geq 3$ *for all* $t\in {'T}$, *then* $\kappa(F_1^*)\geq 3$. *If there is a* $t_0\in {'T}$ *with* $d_{F_1^*}(t_0)\leq 2$, *then* $d_{F_1^*}(t_0)=2, N_{'\overline{F_1}}(t_0)\subseteq N_1$, *and* $\kappa(F_1^*-t_0)\geq 3$.

Proof. Define $d_0:=d_{G_0(T)}(z_0)$ and $\delta_i:=3|F_i^*|-||F_i^*||$ for $i=1,2$. By (3.10) we have $\delta_i\geq 7$ for $i=1,2$. Easy counting gives

$$||G_0||+||G_0(T)||+2(3-d_0)=||F_1^*||+||F_2^*||=3(|G_0|+4)-\delta_1-\delta_2.$$

Using $||G_0||\geq 3|G_0|-6$, this implies

$$\delta_1+\delta_2+||G_0(T)||+2(3-d_0)\leq 18.$$

Since $||G_0(T)||+2(3-d_0)\geq 3$ and $\delta_i\geq 7$, we get $\delta_i=7$ for an $i\in\mathbb{N}_2$ and $\delta_j\leq 8$ for $j\in\mathbb{N}_2-\{i\}$. If $'T$ is not independent or $'T\not\subseteq N_1$, then $||G_0(T)||+2(3-d_0)\geq 4$ and hence $\delta_i=7$ for $i=1,2$. This proves (a).

We assume now $||F_1^*||=3|F_1^*|-7$. Then F_1 is not trivial by (3.3), (3.2) and (3.7). Even $|F_1|\geq 3$ holds by (3.8)(a), (3.3), (3.2), and (3.9). Since F_1^* arises from G_0 by contraction of $G_0(F_2\cup z_0)$ to z_0, we can apply (3.5). As $'T\subseteq N_{F_1^*}(z_0)$, (3.5)(a) says $|N_{F_1^*}(z_0)\cap N_{F_1^*}(t)|\geq 2$ for $t\in {'T}$. Since $[t,t']\in E(G('T))$ implies $t'\in N_1$ by (3.4)(b), $N_{F_1^*}(z_0)\cap N_{F_1^*}(t)\subseteq N_1$, and the second assertion of (b) follows. The last is immediate from (3.5)(b), since $|F_1^*|\geq 7$.

Assume now $||F_1^*||=3|F_1^*|-8$ and $|F_1|\geq 2$. First suppose there is a $t_0\in {'T}$ with $d_{F_1^*}(t_0)\leq 2$. Then $|N_{F_1^*}(t_0)\cap F_1|=1$ by (3.4)(a), say, $s\in N_{F_1^*}(t_0)\cap F_1$. Hence $d_{F_1^*}(t_0)=2$ and F_1 connected by (3.4)(a). Since $|F_1|\geq 2$ and F_1 connected, $S:=(T-\{t_0\})\cup\{s\}\in\mathfrak{T}_0$ and $F:=F_1-s$ is an S-fragment of G_0. Since $d_{F_1^*}(t_0)=2$, we have $||F_1^*-t_0||=3|F_1^*-t_0|-7$. Since $F_1^*-t_0\subseteq F^*$, we conclude $F_1^*-t_0=F^*$ by (3.10), hence $s\in N_1$, and the other claims of (c) but the second follow by application of (b) to F^*.

For the remaining assertion of (c), suppose $\delta(F_1^*)\geq 3$ and the existence of a separating vertex set S in F_1^* with $|S|\leq 2$. Then $z_0\in S$ by (3.4)(a), since $'T\subseteq N_{F_1^*}(z_0)$. Hence there is a $C\in\mathfrak{C}(F_1^*-S)$ with $|C\cap T|\leq 1$. Since $\delta(F_1^*)\geq 3$ implies $V(C)-T\neq\emptyset, S\cup(V(C)\cap T)$ separates G_0, contradicting (3.4)(a). ∎

We can now improve results of (3.4)(b) and (3.8)(b).

(3.12) $'T$ *is independent for every* $T\in\mathfrak{T}_0$.

Proof. Let $'T=\{t_1,t_2,t_3\}$ and assume $[t_1,t_2]\in E(G)$. Then $'T\subseteq N_1$ by (3.4)(b). Let F_1,F_2 be complementary T-fragments of G_0. Choose $z\in H_0('T)$. We may assume that there is an $x\in F_1\cap H_0('T:z)$. By (3.4)(a), we can find an $x,'T$-fan of order 3 in $'G_0$, hence a $\dot{D}(t_1,t_2,x,t_3)\subseteq' \overline{F}_1$. By (3.11)(a), we have $||F_2^*||=3|F_2^*|-7$ and hence $\kappa(F_2^*)\geq 3$ by (3.11)(b). So $\kappa('\overline{F}_2)\geq 2$ and we can find a $t_3,\{t_1,t_2\}$-fan P_1,P_2 in $'\overline{F}_2$. Then $\dot{D}(t_1,t_2,x,t_3)\cup P_1\cup P_2$ is a $\dot{K}_4\subseteq 'G_0$ with $B(\dot{K}_4)='T\cup\{x\}$, which we can expand by an $\mathfrak{F}^x(H_0('T:z))$ to a $\dot{K}_5\subseteq G$. ∎

We prove now the main result of this section.

(3.13) $\tau(H_1)\geq 5$.

Proof. Suppose H_1 contains a circuit of length at most 4, hence a quadrangle Q by (3.7); say, $V(Q)=\{q_i : i\in\mathbb{N}_4\}$. Choose $z\in H_0(q_1,q_2,q_3)$. If $q_4\in H_0(q_1,q_2,q_3:z)$, we find $x\in N_1$ with $\{x,q_4\}\in H_0(q_1,q_2,q_3:z)$ by (3.6). Then an $x,V(Q)$-fan of order 3 in $'G_0$ (by (3.4)(a)) would furnish a $\dot{K}_4\subseteq 'G_0$ with $B(\dot{K}_4)\subseteq\{x\}\cup V(Q)$, and we would get a $\dot{K}_5\subseteq G$. This shows $q_4\notin H_0(q_1,q_2,q_3:z)$.

Consider $\{x_1,x_2\}\in H_0(q_1,q_2,q_3:z)$. As we have seen, $q_4\neq x_1,x_2$. There is a z,N_1-fan $P_1,\dots,P_5$ of order 5 in G ending in q_1,q_2,q_3,x_1,x_2; set $F:=\bigcup_{i=1}^{5} P_i - N_1$. Since H_0 is connected and $q_4\in N_1$, there is a q_4,F-path in $G(H_0\cup q_4)$, say, a q_4,y-path. Since $y\in P_4\cup P_5$ would imply the contradiction $q_4\in H_0(q_1,q_2,q_3:z)$, we conclude $y\in\bigcup_{i=1}^{3} P_i - z$, say, $y\in P_1 - z$. This implies $z\in H_0(q_2,q_3,q_4)$ and $\{x_1,x_2\}\in H_0(q_2,q_3,q_4:z)$.

Suppose there is a q_2,q_3-path containing an $x_i (i\in\mathbb{N}_2)$ in $G_0-\{z_0,q_1,q_4\}$. Then by (3.4)(a) and Theorem P, there is an x_i,Q-fan of order 3 in G_0-z_0 ending in q_2,q_3,q_j for a $j\in\{1,4\}$. In both the cases $j=1$ and $j=4$, we would get a $\dot{K}_4\subseteq 'G_0$, where $B(\dot{K}_4)$ are the endvertices of a z,N_1-fan of order 4 in G. This contradiction shows that there is a q_2,q_3-path P in the connected $G_0-\{z_0,q_1,q_4\}$ with $V(P)\cap\{x_1,x_2\}=\emptyset$ and that for $i=1,2$, there is a vertex s_i separating x_i from $\{q_2,q_3\}$ in $G_0-\{z_0,q_1,q_4\}$. Then $S_i:=\{z_0,q_1,q_4,s_i\}\in\mathfrak{T}_0$ for $i=1,2$, hence $\{q_1,q_4,s_i\}$ independent by (3.12). So $s_i\notin Q$ and q_1,q_4, hence also q_2,q_3 are opposite in the quadrangle Q. We may assume $d:=|N(x_1)\cap\{q_1,q_4\}|\leq|N(x_2)\cap\{q_1,q_4\}|$.

If $d=2$, using Q,P, and the path q_1,x_1,q_4, we get a $\dot{K}_4\subseteq G_0-\{z_0,x_2\}$ with $B(\dot{K}_4)=V(Q)$. But then we could enlarge a z,N_1-fan of order 4 ending in q_1,q_2,q_3,x_2 by $[x_2,q_4]$, to get a $\dot{K}_5\subseteq G$. So we must have $d\leq 1$.

Consider $C\in\mathfrak{C}(G_0-S_1)$ with $x_1\in C$. Since $d\leq 1, |C|\geq 2$. By (3.10) and (3.11)(a), $3|C^*|-7\geq||C^*||\geq 3|C^*|-8$ holds. By (3.11)(b) and (c), we may assume $d_{C^*}(q_1)\geq 3$. Since $s_1\notin Q, q_2,q_3\in\widetilde{C}$. Since q_2,S_1 have the property of y,T in (3.9), we get $c(G_0-S_1)=2$, and $\widetilde{C}$ is connected. Hence, there is a q_2,q_3-path $P'\subseteq\widetilde{C}$ and $V(P')\cap\{x_1,x_2\}=\emptyset$, as we have seen above. If there is a $q_1,\{q_4,x_i\}$-fan Q_1,Q_2 in $'\overline{C}$ ending in q_4,x_i for an $i\in\mathbb{N}_2$, then Q,Q_1 and P' form a $\dot{K}_4\subseteq 'G_0$ with $B(\dot{K}_4)=V(Q)$ and we can extend a z,N_1-fan of order 4 ending in q_2,q_3,q_4,x_i by

Q_2 to get a $\dot{K}_5 \subseteq G$. So there is not such a $q_1,\{q_4,x_i\}$-fan, neither for $i=1$ nor for $i=2$. For $x_1 \in C$, this means by (3.11)(b) and (c), that the case (3.11)(c) occurs for $C^*, d_{'\overline{C}}(q_4)=1$ and $[q_4,x_1] \in E(\overline{C})$. Then by (3.11)(c) again, $x_2 \notin \overline{C}$. Since $q_4 \in N(x_1), d=1$ and $N(x_2)\cap\{q_1,q_4\} \neq \emptyset$ by the choice of x_1, say, $q_j \in N(x_2)$ for a $j \in \{1,4\}$. Then Q, P', and a q_1,q_4-path in $'\overline{C}$ form a $\dot{K}_4 \subseteq G_0 - \{z_0,x_2\}$ with $B(\dot{K}_4)=V(Q)$ and for the $j' \in \{1,4\}-\{j\}$ we can enlarge a z,N_1-fan of order 4 ending in $q_2,q_3,q_{j'},x_2$ by $[x_2,q_j]$ to get a $\dot{K}_5 \subseteq G$. ∎

For later use, we define $H_2 := {'G_0} - N_1$ and note

(3.14) *Every $F \subseteq G_0$ with $z_0 \in F, ||F|| \geq 3|F|-7$, and $|F| \geq 5$ contains a vertex of H_2. In particular, $|H_2| \geq 1$.*

Proof. If $'F \subseteq H_1$, then (3.13) and Corollary M imply the existence of a $\dot{K}_5 \subseteq G$, for $'F$ isomorphic to the pentagon or the Petersen graph cannot happen, since otherwise $||F|| \leq 3|F|-8$. ∎

(3.15) *For every $x \in N_1, H_1 - x$ does not contain the Petersen graph.*

Proof. Assume, for an $x_0 \in N_1$, there is a $P \subseteq H_1 - x_0$ isomorphic to the Petersen graph. By (3.4)(a), there is an x_0,P-fan $\mathfrak{F}_3$ of order 3 in $'G_0$ ending in a_1,a_2,a_3, say. By lemma Pet, there is a $\dot{K}_4 \subseteq P$ with $\{a_1,a_2,a_3\} \subseteq B(\dot{K}_4)$, say, $B(\dot{K}_4)=\{a_1,a_2,a_3,a_4\}$. Then $\dot{K}_4, \mathfrak{F}_3$, and a path $x_0H_0a_4$ form a $\dot{K}_5$ with $B(\dot{K}_5)=B(\dot{K}_4)\cup\{x_0\}$. ∎

Furthermore, we will need the following special result related to (3.11)(b).

(3.16) *Let F be a T-fragment of G_0 for a $T \in \mathfrak{T}_0$ with $||F^*|| = 3|F^*|-7$. Then $\kappa((F^*-t_1)-[z_0,t_2]) \geq 2$ for all $\{t_1,t_2\} \in \mathfrak{P}_2('T)$.*

Proof. We have $|N_{G_0}(t)\cap F| \geq 2$ for all $t \in {'T}$ by (3.11)(b) and (3.12). This also holds for $t=z_0$. If $F \cap H_2 = \emptyset$, this follows from (3.11)(b). If $F \cap H_2 \neq \emptyset$, say, $b \in F \cap H_2$, there is a b,N_1-fan of order 5 in G, since $\kappa(G) \geq 5$, hence a b,N_1-fan of order 2 in $G_0 - T$, which implies $|N_1 \cap F| \geq 2$.

Suppose there is a separating vertex s in $(F^*-t_1)-[z_0,t_2]$. Then $S := \{s,t_1\}$ separates $F^*-[z_0,t_2]$ and z_0 and t_2 are in different components C_1 and C_2 of $(F^*-[z_0,t_2])-S$ by (3.11)(b). Then $|C_i| \geq 2$ for $i=1,2$, since $|N_{G_0}(t)\cap F| \geq 2$ for $t \in T$ and $|S \cap F| \leq 1$. Since $|T \cap C_{i_0}|=1$ for an $i_0 \in \mathbb{N}_2$, we get a separating set $S \cup (T \cap V(C_{i_0}))$ of G_0 consisting of three vertices, contradicting (3.4)(a). ∎

4. Completion of the proof of Theorem 2

In this section we will finish the proof of Theorem 2. The main tool in the proof will be Theorem M, which we shall apply to an appropriate subgraph of H_1 in the form of Corollary M.

Let us first assume that G_0 is not 5-connected, hence $\mathfrak{T}_0 \neq \emptyset$. Consider $\mathfrak{U}_0 := \{U : U$ *non-trivial fragment of* $G_0 - T$ *for a* $T \in \mathfrak{T}_0\} \neq \emptyset$ and choose $U_0 \in \mathfrak{U}_0$ such that $|U_0| = \min\{|U| : U \in \mathfrak{U}_0\}$. Set $T_0 := N_{G_0}(U_0) \in \mathfrak{T}_0$. Then $z_0 \in T_0$ by (3.4)(b). We define $\overline{U}_0 := G(U_0 \cup {}'T_0)$, the *graph* $U_0^* := G_0(U_0 \cup T_0) \cup \{[z_0, t] : t \in {}'T_0\}$, and $U_0^+ := U_0^* \cup \{[t,t'] : \{t,t'\} \in \mathfrak{P}_2({}'T_0)\}$ (cf.(3.12)).

If G_0 is 5-connected, we define these graphs in the following way. Since a $\dot{K}_5^- \subseteq H_1$ can be extended to a $\dot{K}_5 \subseteq G$ by a suitable path xH_0y, there is no subdivision of K_5^- in H_1. Hence, by (3.2) and the result of J. Pelikán [8] that $\delta(H_1) \geq 4$ implies the existence of a $\dot{K}_5^- \subseteq H_1$ or by (3.2), (3.13), and Theorem M, H_1 contains a vertex v_0 with $d_{H_1}(v_0) = 3$. Now we define $T_0 := N_{H_1}(v_0) \cup \{z_0\}$, $U_0 := G_0 - T_0, \overline{U}_0 := {}'G_0$, and $U_0^* := U_0^+ := G_0$.

We prove some properties of U_0, T_0.

(4.1) (a) U_0 *is connected;*

(b) $|U_0| \geq 3$*;*

(c) $|N(t) \cap U_0| \geq 2$ *for all* $t \in T_0$*;*

(d) $C \cap N_1 \neq \emptyset$ *for all* $C \in \mathfrak{C}(G_0 - T_0)$ *and* $|U_0 \cap N_1| \geq 2$.

Proof. Using (3.13), these properties are immediate, if G_0 is 5-connected. So assume $T_0 \in \mathfrak{T}_0$. If U_0 is disconnected, by the minimal choice of U_0, it consists of two singletons x_1 and x_2 with $d_{G_0}(x_1) = d_{G_0}(x_2) = 4$. But this implies $\{x_1, x_2\} \subseteq N_1$, hence ${}'T_0 \subseteq N_1$ by (3.2) and so the existence of a quadrangle in H_1, a contradiction to (3.13). So (a) follows and similarly we get (b) from (3.2) and (3.13), since $U_0 \cap H_2 \neq \emptyset$ is impossible for $|U_0| = 2$ by (3.3), since $z_0 \in T_0$. Then (b) and the minimum property of U_0 imply (c). If for a $C \in \mathfrak{C}(G_0 - T_0), C \cap H_2 \neq \emptyset$, then ${}'T_0 \cup (V(C) \cap N_1)$ separates G and so $|C \cap N_1| \geq 2$, since $\kappa(G) \geq 5$. This implies (d). ∎

(4.2) (a) U_0^+ *is* 4^+*-connected;*

(b) *For every* $D \subseteq \overline{U_0}$ *with* $|D| \geq 5$ *and* $x \in (H_2 \cap \overline{U}_0) - V(D)$*, there is an* x, D*-fan of order* 4 *in* ${}'G_0$.

Proof. We may assume again $\kappa(G_0) = 4$. Suppose there is a $T \subseteq V(U_0^+)$ with $|T| \leq 4$ separating U_0^+. Since T_0 spans a K_4 in U_0^+, there is a T-fragment F of $U_0^+ - T$ with $F \cap T_0 = \emptyset$. Hence $T \in \mathfrak{T}_0$ and $V(F) \subsetneqq V(U_0)$ by (4.1)(a), so $|F| = 1$ and $|\mathfrak{C}(U_0^+ - T)| = 2$ by the minimal choice of U_0. This proves (a). Let D and x be as described in (b). Since ${}'U_0^+$ is 3^+-connected by (a), there is an x, D-fan of order 4 in $'U_0^+$ by (3.3),(4.1)(c), and lemma M (a). Choose such an x,D-fan P_1,P_2,P_3,P_4 containing a minimum number of edges from $E(U_0^+('T_0))$. If there is at most one edge of $E(U_0^+('T_0))$ contained in $\bigcup_{i=1}^{4} P_i$, then we can replace this edge with a path in the connected graph $G(\widetilde{U}_0 \cup 'T_0)$ to get an x,D-fan of order 4 in $'G_0$. So we may assume that there are exactly two edges e_1,e_2 of $E(U_0^+('T_0))$ in $\bigcup_{i=1}^{4} P_i$. Then e_1,e_2 are incident to x by the minimal choice of the fan, in particular, $x \in 'T_0$. Then $'T_0 \not\subseteq N_1$ and so $||\widetilde{U}_0^*|| = 3|\widetilde{U}_0^*| - 7$ by (3.11)(a) and $\kappa('\widetilde{U}_0^*) \geq 2$ by (3.11)(b), where $\widetilde{U}_0^*$ means $(\widetilde{U}_0)^*$. But then we can replace e_1,e_2 with an $x,('T_0 - \{x\})$-fan of order 2 in $'\widetilde{U}_0^* = \overline{\widetilde{U}}_0$ to get an x,D-fan of order 4 in $'G_0$. ∎

Now consider $\mathbb{S}_0 := \{S \subseteq N_1 : |S| \leq 3$ *and* $H_1 - S$ *has a non-trivial fragment* $F \subseteq U_0\}$. Then $T_0 \cap N_1 \in \mathbb{S}_0$ by (4.1)(d) or by definition of T_0, (3.2), and (3.13). So we can choose $S_0 \in \mathbb{S}_0$ and a non-trivial fragment $F_0 \subseteq U_0$ of $H_1 - S_0$ such that $|F_0| = \min\{|F| : F \subseteq U_0$ *is a non-trivial fragment of* $H_1 - S$ *for some* $S \in \mathbb{S}_0\}$. First we prove corresponding properties for F_0 as in (4.1) and (4.2) for U_0.

(4.3) (a) F_0 *is connected;*
(b) $|F_0| \geq 3$*;*
(c) $|S_0| = 3$ *and* $N_{H_1}(F_0) = S_0$*;*
(d) $|N(s) \cap F_0| \geq 2$ *for all* $s \in S_0$.

Proof. (a) follows from (3.2), (3.13), and the minimal choice of F_0. Again by (3.2) and (3.13), (b) follows and then (c) and (d) by (b) and the minimum property of F_0. ∎

Using the notation $\overline{F}_0 := H_1(F_0 \cup S_0), \overline{F}_0 \subseteq \overline{U}_0$ by (4.3)(c) and then $\delta(\overline{F}_0) \geq 2$ and $V_2(\overline{F}_0) \subseteq S_0$ by (3.2) and (4.3)(d). $\overline{F}_0$ is connected by (4.3) (a)/(d), but one can easily prove more.

(4.4) $\overline{F}_0$ *is* 2^+*-connected.*

Proof. Assume $T \subseteq V(\overline{F}_0)$ separates $\overline{F}_0$ and $|T| \leq 2$. Then there is a $C \in \mathfrak{C}(\overline{F}_0 - T)$ with $|C \cap S_0| \leq 1$. Since $T \cap F_0 \neq \emptyset$ by (4.3)(a)/(d), $|C - S_0| < |F_0|$. $|C - S_0| = 1$ is impossible, since this would imply $|C \cap S_0| = 1$ and hence the existence of a triangle in $H_1(C \cup T)$ by (3.2) and (4.3)(d), contradicting (3.13). So $|C - S_0| \geq 2$ or $|C - S_0| = 0$. If $|C - S_0| \geq 2$, then for $T' := T \cup (V(C) \cap S_0) \in \mathfrak{P}_{\leq 3}(N_1), H_1 - T'$ would have a smaller non-trivial fragment $C - S_0 \subseteq F_0 \subseteq U_0$ than S_0 has, a contradiction to the choice of F_0, S_0. So we conclude that C consists of a vertex $s \in S_0$, hence $T \subseteq N(s)$ and $|T| = 2$ by (4.3)(d). Hence $\kappa(\overline{F}_0) \geq 2$ and by (3.13) there is exactly one $C \in \mathfrak{C}(\overline{F}_0 - T)$ with $|C \cap S_0| \leq 1$. This shows that $\overline{F}_0$ is 2^+-connected. ∎

In some cases we will have to consider $\overline{F}_0$ and a path connecting two vertices of $\overline{F}_0$. So let x,y be distinct vertices of $\overline{F}_0$ and let $P_0 \subseteq {}'G_0$ be an x,y-path with $V(P_0)\cap V(\overline{F}_0)=\{x,y\}$. Then we define the *graph* $\overline{F}_0^{P_0} := \overline{F}_0 \cup [x,y]$ and for every $F \subseteq \overline{F}_0^{P_0}$, we denote $F_{P_0} := (F-[x,y]) \cup P_0$, if $[x,y] \in E(F)$, and $F_{P_0} := F$, if $[x,y] \notin E(F)$. Similarly, we define the *graph* $G_0^{P_0} := (G_0 - V(\mathring{P}_0)) \cup [x,y]$ and for $H \subseteq G_0^{P_0}, H_{P_0} := (H-[x,y]) \cup P_0$ or $H_{P_0} := H$, if $[x,y] \in E(H)$ or $[x,y] \notin E(H)$, respectively. We use the same notation also when P_0 denotes the empty graph: then $\overline{F}_0^{P_0} := \overline{F}_0$, $F_{P_0} := F$, $G_0^{P_0} := G_0$, and $H_{P_0} := H$.

(4.4), (4.3)(b)/(c), and lemma M (b) imply $\kappa(x,y;\overline{F}_0) \geq 3$ for all $x \neq y$ from $V_{\geq 3}(\overline{F}_0)$. Since $\overline{F}_0^{P_0}$ is 2^+-connected, too, $\kappa(x,y;\overline{F}_0^{P_0}) \geq 3$ for all $x \neq y$ from $V_{\geq 3}(\overline{F}_0^{P_0})$ follows in the same way.

Now we will consider $G_1 := G_0 - V(\overline{F}_0)$ and $\overline{G}_1 := G_0(G_1 \cup S_0) = G_0 - V(F_0)$. Our next aim is to show that ${}'G_1$ is connected and $|N_G({}'G_1) \cap F_0| = 2$. For this we need some preparation.

(4.5) *Let c_1,c_2 be distinct vertices of $\overline{F}_0$ with $\{c_1,c_2\} \cap S_0 \neq \emptyset$ and let P_0 be a c_1,c_2-path in ${}'G_0$ with $\mathring{P}_0 \subseteq {}'G_1$ or $P_0 = \emptyset$. Set $F := \overline{F}_0^{P_0}$. Assume P_1,P_2,P_3 are openly disjoint a_1,a_2-paths in F for distinct $a_1,a_2 \in F$. Then the following statements hold.*

(a) *If there is an $x, \bigcup_{i=1}^{3} P_i$-fan Q_1,Q_2,Q_3,Q_4 in ${}'G_0^{P_0}$ ending in a_1,a_2,a_3,a_4 for an $x \in {}'G_0^{P_0} - \bigcup_{j=1}^{3} V(P_j)$, then $a_3,a_4 \in P_i$ for an $i \in \mathbb{N}_3$.*

(b) *If for an $x \in {}'G_1 - V(P_0)$, there is an x,F-fan Q_1,Q_2,Q_3,Q_4 in ${}'G_0^{P_0}$ ending in a_1,a_2,a_3,a_4 with $a_3 \in P_1$, then $d_F(a_3)=2$.*

(c) *Assume for an $a_3 \in F - \bigcup_{j=1}^{3} V(P_j)$, there is an $a_3, \bigcup_{i=1}^{3} P_i$-fan L_1,L_2,L_3 in F ending in b_1,b_2,b_3 with $b_i \in \mathring{P}_i$ for $i \in \mathbb{N}_3$ and set $D := \bigcup_{i=1}^{3} (P_i \cup L_i) \subseteq F$. Then for no $x \in {}'G_0^{P_0} - V(D)$, an x,D-fan of order 4 ending in a_1,a_2,a_3,a_4 exists in ${}'G_0^{P_0}$.*

Proof. (a) Assume there is such an $x, \bigcup_{i=1}^{3} P_i$-fan $Q_j (j \in \mathbb{N}_4)$ with $a_{i+2} \in P_i$ for each $i \in \mathbb{N}_2$, say. Since $\left(\bigcup_{i=1}^{3} P_i\right)_{P_0}$ forms a $\dot{K}_4^- \subseteq {}'G_0$ with

$$B(\dot{K}_4^-) = \{a_1,a_2,a_3,a_4\} \subseteq N_1,$$

$\left(\bigcup_{j=1}^{4} Q_j \cup \bigcup_{i=1}^{3} P_i\right)_{P_0}$ forms a $\dot{K}_5^- \subseteq {}'G_0$ with $B(\dot{K}_5^-) = B(\dot{K}_4^-) \cup \{x\}$ and hence $\dot{K}_5^- \cup a_3 H_0 a_4$ is a $\dot{K}_5 \subseteq G$.

(b) Suppose $d_F(a_3) \geq 3$. If $|P_2 \cup P_3| = 3$, say, $V(P_2 \cup P_3) = \{a_1, a_2, b\}$, then $P_2 \cup P_3 \not\subseteq \overline{F}_0$ by (3.13), hence $[c_1, c_2] \in E(P_2 \cup P_3) - E(\overline{F}_0)$. But this implies $d_F(b) \geq 3$, since then $d_F(c_i) \geq 3$ for $i = 1,2$ and in the case $\{c_1, c_2\} = \{a_1, a_2\}$, we have $\{a_1, a_2\} \cap S_0 \neq \emptyset$ by assumption on $\{c_1, c_2\}$, so $b \in F_0$ or $d_{F(S_0)}(b) \geq 1$, hence $d_F(b) \geq 3$ by (4.3)(d). Therefore, by lemma M(a), an $a_3,(P_2 \cup P_3)$-fan of order 3 exists, since F is 2^+-connected by (4.4). Hence by Theorem P, we find an $a_3,(P_2 \cup P_3)$-fan L_1, L_2, L_3 in F ending in a_1, a_2, a. We may assume, $a \in P_2$, and denote $D := P_2 \cup P_3 \cup \bigcup_{i=1}^{3} L_i$. If there is an a_4,D-path Q in F ending in $a' \in D - \{a_1, a_2, a_3\}$, then the x,D-fan $Q_1, Q_2, Q_3, Q_4 \cup Q$ in ${}'G_0^{P_0}$ together with at least one of the following three triples of openly disjoint a_i,a_j-paths ($1 \leq i < j \leq 3$) in D contradicts (a), namely the triple $L_1 \cup L_2, P_2, P_3$ (if $a' \in P_3$) or $L_1, P_2[a_1,a] \cup L_3, P_3 \cup L_2$ (if $a' \in L_1 \cup P_2[a_1,a] \cup L_3$) or $L_2, P_2[a_2,a] \cup L_3, P_3 \cup L_1$ (if $a' \in L_2 \cup P_2[a_2,a] \cup L_3$) (see figure 4.1). This contradiction shows $a_4 \notin D$ and the existence of an a_4,D-fan L_1', L_2' of order 2 in F ending in a_i, a_j with $i,j \in \mathbb{N}_3$, because $\kappa(F) \geq 2$ by (4.4). But then Q_1, Q_2, Q_3, Q_4 and three openly disjoint a_i,a_j-paths in F, namely $L_1' \cup L_2'$ and two appropriate a_i,a_j-paths in D, give a contradiction to (a).

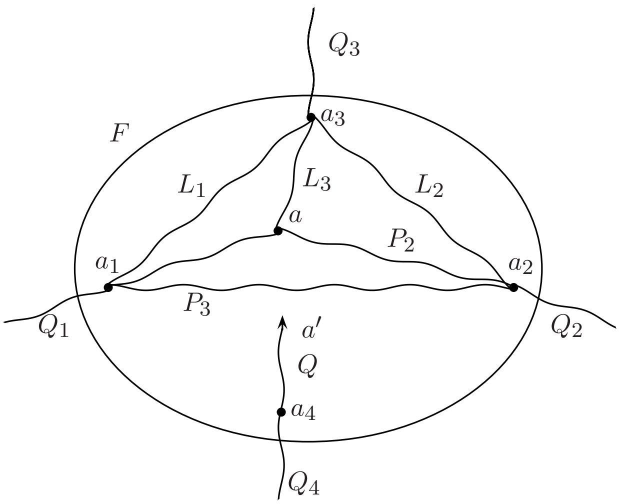

Fig. 4.1

(c) Assume, for an $x \in {}'G_o^{P_0} - V(D)$, there is an x,D-fan Q_1,Q_2,Q_3,Q_4 in ${}'G_0^{P_0}$ ending in a_1,a_2,a_3,a_4. By symmetry, we may assume $a_4 \in \bigcup_{i=1}^{3} P_i$ and $a_4 \notin P_1$. Then $Q_1,Q_2,Q_3 \cup L_1,Q_4$ together with P_1,P_2,P_3 contradict (a). ∎

(4.6) Proposition. *Let c_1,c_2 be distinct vertices of $\overline{F}_0$ with $\{c_1,c_2\} \cap S_0 \neq \emptyset$ and let P_0 be a c_1,c_2-path in ${}'G_0$ with $\overset{\circ}{P}_0 \subseteq {}'G_1$ or $P_0 = \emptyset$. Set $F := \overline{F}_0^{P_0}$.*

Then there is no $x \in {}'G_1 - V(P_0)$ such that there is an x,F-fan Q_1,Q_2,Q_3,Q_4 in ${}'G_0^{P_0}$ ending in a_1,a_2,a_3,a_4 with $\{a_1,a_2,a_3\} \subseteq V_{\geq 3}(F)$.

Proof. Suppose there is such an x,F-fan Q_1,Q_2,Q_3,Q_4 for an $x \in {}'G_1 - V(P_0)$. Then there are openly disjoint a_1,a_2-paths P_1,P_2,P_3 in F by (4.3)(b)/(c) and lemma M(b), since F is 2^+-connected by (4.4). From (4.5)(b) we can conclude $a_3 \notin \bigcup_{i=1}^{3} P_i$, since $d_F(a_3) \geq 3$. Since $|\bigcup_{i=1}^{3} P_i| \geq 4$ and $a_3 \in V_{\geq 3}(F)$, by (4.4) and lemma M (a), there is an $a_3, \bigcup_{i=1}^{3} P_i$-fan L_1,L_2,L_3 of order 3 in F ending in b_1,b_2,b_3. Assume there is an $i \in \mathbb{N}_3$ with $|V(P_i) \cap \{b_1,b_2,b_3\}| \geq 2$, say, $b_1,b_2 \in P_3$ with $b_1 \in P_3[a_1,b_2]$. Then the openly disjoint a_1,a_2-paths $P_1,P_2,P_3[a_1,b_1] \cup L_1 \cup L_2 \cup P_3[b_2,a_2]$ contradict (4.5)(b), since $a_3 \in V_{\geq 3}(F)$ is on the last path. This contradiction shows that we may assume $b_i \in \overset{\circ}{P}_i$ for all $i \in \mathbb{N}_3$. Defining $D := \bigcup_{i=1}^{3} (P_i \cup L_i)$, we can apply (4.5)(c) and conclude that there is no a_4,D-path Q in F with endvertex in $D - \{a_1,a_2,a_3\}$, since otherwise $Q_1,Q_2,Q_3,Q_4 \cup Q$ would contradict (4.5)(c). In particular $a_4 \notin D$. Hence there is an a_4,D-fan L'_1,L'_2 of order 2 in F ending in a_i,a_j with $i,j \in \mathbb{N}_3$, since $\kappa(F) \geq 2$ by (4.4); say, $i=1$ and $j=3$. Then the openly disjoint a_1,a_3-paths $L'_1 \cup L'_2, P_1[a_1,b_1] \cup L_1, P_2 \cup P_3[a_2,b_3] \cup L_3$ in F contradict (4.5)(a). ∎

We use the following notation for the rest of the proof. Define $A := N_G({}'G_1) \cap V(F_0)$ and $B := N_G(A) \cap V({}'G_1)$. Since F_0 is a component of $H_1 - S_0$ and $S_0 \cap G_1 = \emptyset$, we see $B \subseteq V(H_2)$. Since $F_0 \subseteq U_0$, also $B \subseteq \overline{U}_0$. Furthermore, we define a bipartite graph $G(A,B) := (A \cup B, [A,B]_G) \subseteq G$, $\triangle := \sum_{b \in B} (d_{G(A,B)}(b) - 1) = ||G(A,B)|| - |B|$, and $G_1^* := \overline{G}_1 \cup \{[z_0,b] : b \in B\}$. We note some properties of G_1^*.

(4.7) (a) $||G(A,B)|| = ||G_1^*|| - ||\overline{G}_1|| + \triangle$;

(b) $||G_1^*|| \leq 3|G_1^*| - 7$;

(c) $\kappa(G_1^*) \geq 3$, *if equality holds in (b).*

Proof. Since $B \subseteq V(H_2), ||G_1^*|| - ||\overline{G}_1|| = |B|$, and (a) follows from the definition of $\triangle$. Since G_1^* arises from G_0 by contracting the connected subgraph $G_0(F_0 \cup z_0)$ to z_0, (b) follows from (3.1). Since $H_2 \neq \emptyset$ by (3.14), $H_1 - V(\overline{F}_0) \neq \emptyset$ and $|S_0| = 3$ by (4.3)(c), we can apply (3.5), if equality in (b) holds, and get $\kappa(G_1^*) \geq 3$ or $'\overline{G}_1$ consists of two triangles sharing exactly one vertex c. But the latter case cannot occur, since otherwise $|H_2| = |H_1 - V(\overline{F}_0)| = 1$ and $c \in H_1 - V(\overline{F}_0)$ by (3.2), since c is the only vertex of degree exceeding 2 in $'\overline{G}_1$. Hence one of the two triangles of $'\overline{G}_1$ is in H_1, a contradiction to (3.13). ∎

(4.8) $\triangle \geq 1$ *and* $|A| \geq 2$. *If* $\triangle = 1$, *then* $||G_1^*|| = 3|G_1^*| - 7$.

Proof. Since $\overline{F}_0 \subseteq H_1, ||G_0(\overline{F}_0 \cup z_0)|| \leq 3|G_0(\overline{F}_0 \cup z_0)| - 9$ follows from (3.13) and Corollary M, since $|\overline{F}_0| \geq 6$ by (4.3) and $\overline{F}_0$ is not the Petersen graph by (3.15) and since a $\dot{K}_5 \subseteq G_0(\overline{F}_0 \cup z_0)$ with $z_0 \notin B(\dot{K}_5)$ gives a subdivision of K_5 in G replacing z_0 with a path through H_0. By addition of this inequality and (4.7)(b), we get by (4.7)(a) $||G_0|| - \triangle + 3 + ||G(S_0)|| = ||G_0(\overline{F}_0 \cup z_0)|| + ||G_1^*|| \leq 3(|G_0| + 4) - 16$, since $S_0 \subseteq N_1$. Using $||G_0|| \geq 3|G_0| - 6$, this implies $\triangle \geq 1$. If equality holds here, it must also hold in the inequalities added, especially in (4.7)(b). But $\triangle \geq 1$ means that there is a $b \in B$ with $d_{G(A,B)}(b) \geq 2$, which implies $|A| \geq 2$. ∎

(4.9) Proposition. $'G_1$ *is connected.*

Proof. Suppose $c('G_1) \geq 2$. $B \neq \emptyset$ by (4.8), say, $x \in B$. Then $x \in H_2 \cap \overline{U}_0$ by definition of F_0, S_0. Consider $C \in \mathfrak{C}('G_1)$ containing x. By assumption, there is a $C' \neq C$ in $\mathfrak{C}('G_1)$. First we show

$$|N_G(C') \cap S_0| \geq 2. \tag{4.9.1}$$

Proof. If $N_G(C') \cap F_0 \neq \emptyset$, then there is a $y \in C' \cap H_2 \cap \overline{U}_0$. Hence there is a $y, \overline{F}_0$-fan $\mathfrak{F}_4$ of order 4 in $'G_0$ by (4.3) and (4.2)(b) ending in $Y \in \mathfrak{P}_4(V(\overline{F}_0))$. Then (4.6) (for $P_0 = \emptyset$) implies $|Y \cap V_{\geq 3}(\overline{F}_0)| \leq 2$, hence $|Y \cap S_0| \geq 2$ by (3.2). Since $Y \subseteq N_G(C')$, we may therefore assume, $N_G(C') \cap F_0 = \emptyset$. But this means $S_0 \cup \{z_0\} \in \mathfrak{T}_0$ and hence $S_0 \cup \{z_0\} = N_G(C')$ by (3.4) (a) and (4.3)(c). ∎

There are $c_1 \neq c_2$ in $N_G(C') \cap S_0$ by (4.9.1). Hence there is a c_1, c_2-path P_0 in $G(C' \cup \{c_1, c_2\})$. Define $F := \overline{F}_0^{P_0}$. Then $d_F(c_i) \geq 3$ for $i = 1,2$ by (4.3)(d). Since $x \in H_2 \cap \overline{U}_0$, by (4.2)(b) again, there is an $x, \overline{F}_0$-fan $\mathfrak{F}_4$ of order 4 in $'G_0$ ending in $X \in \mathfrak{P}_4(V(F))$. Since $\overset{\circ}{P}_0 \subseteq C'$ and $x \in C$, hence $\mathfrak{F}_4$ in $'G_0^{P_0}$, (4.6) implies again $|X \cap V_{\geq 3}(F)| \leq 2$, hence $|X \cap V_2(F)| \geq 2$, which is impossible, since $|V_2(F)| \leq |S_0 - \{c_1, c_2\}| = 1$ by (4.3). ∎

Now we can easily determine $|A|$.

(4.10) Proposition. $|A|=2$.

Proof. Suppose $|A|\geq 3$. Since $\triangle\geq 1$ by (4.8), there is a $b\in B$ with $|N(b)\cap A|\geq 2$, say, $\{a_1,a_2\}\subseteq N(b)\cap A$. Since $|A|\geq 3$, there is an $a_3\in A-\{a_1,a_2\}$. Since $'G_1$ is connected by (4.9), there is a b,a_3-path in $G('G_1\cup a_3)$. Since $b\in H_2\cap\overline{U}_0$, there is a $b,\overline{F}_0$-fan of order 4 in $'G_0$ by (4.2)(b), hence also one ending in a_1,a_2,a_3,a_4 by Theorem P. But since $\{a_1,a_2,a_3\}\subseteq V_{\geq 3}(\overline{F}_0)$, this contradicts (4.6) for $P_0=\emptyset$. So $|A|\leq 2$, hence equality by (4.8). ∎

We complete now the proof distinguishing the cases $\triangle\geq 2$ and $\triangle=1$ according to (4.8).

(i) $\triangle\geq 2$.

Then, by definition of $\triangle$ and (4.10), there are $b_1\neq b_2$ in B with $A\subseteq N(b_i)$ for $i=1,2$. Let $A=\{a_1,a_2\}$ by (4.10). There are openly disjoint a_1,a_2-paths P_1,P_2,P_3 in $\overline{F}_0$ by (4.4), lemma M(b), and (4.3)(b)/(c). Set $V_0:=(V(P_1\cup P_2\cup P_3)-A)\cup\{z_0\}$. First we show

(4.11) *There are $p_1\neq p_2$ in V_0 such that there is a p_1,p_2-path P in $G_0-((V_0\cup A)-\{p_1,p_2\})$ containing b_1 and b_2.*

Proof. First we show that *such a p_1,p_2-path P' exists in U_0^+*. (Note $V_0\subseteq\overline{F}_0\cup B\cup\{z_0\}\subseteq U_0^+$.) Then we shall use P' to verify (4.11).

Since $B\subseteq V(H_2)$, we have $d_{U_0^+}(b)\geq 5$ for all $b\in B$ by (4.1)(c) and definition of U_0^+. Therefore, there is a $b_1,V_0\cup\{b_2\}$-fan of order 3 in U_0^+-A by lemma M (a), since U_0^+-A is 2^+-connected by (4.2)(a) and $|V_0|\geq 3$. Since $'G_1$ is connected by (4.9), $'U_0^+-V(\overline{F}_0)$ is connected, too. So there is a b_1,b_2-path in $'U_0^+-V(\overline{F}_0)$, hence, there is also a $b_1,V_0\cup\{b_2\}$-fan Q_1,Q_2,Q_3 in U_0^+-A ending in q_1,q_2,b_2 by Theorem P. Again by (4.2)(a), lemma M(a), and Theorem P, there is a b_2,V_1-fan R_1,R_2,R_3 in U_0^+-A ending in r_1,r_2,b_1 for $V_1:=V_0\cup V(Q_1\cup Q_2)$. If $(Q_1\cup Q_2)\cap(R_1\cup R_2)=\emptyset$, we can take, for instance, the path $P':=Q_1\cup R_3\cup R_1$. So we may assume $Q_1\cap R_1\neq\emptyset$, hence $r_1\in Q_1-b_1$. Then the path $P':=Q_2\cup R_3\cup R_1\cup Q_1[r_1,q_1]$ has the property wanted.

Say, P' is a p_1,p_2-path in U_0^+ as constructed in the preceding paragraph. Now P' can be modified to a p_1,p_2-path P in $G_0-((V_0\cup A)-\{p_1,p_2\})$ with the required property. For this, we have to replace in P' these edges of $E_0:=E(P')\cap E(U_0^+(T_0))$ which do not belong to G_0. We may assume that $\{[x,y],[y,z]\}\in\mathfrak{P}_2(E_0)$ implies $y\in\{b_1,b_2\}$, since $U_0^+(T_0)\cong K_4$. Since $|\{b_1,b_2\}\cap T_0|\leq 1$ by (3.4)(b), this means,

(i) $|E_0|\leq 1$ or

(ii) E_0 consists of two edges $[t_1,b]\neq[t_2,b]$ with $b\in\{b_1,b_2\}$ or

(iii) E_0 consists of two disjoint edges $[t_1,t_2]$ and $[z_0,t]$.

If case (i) occurs, then we can take $P := P'$ or get P from P', replacing the edge $[x,y] \in E_0$ with an x,y-path in $G_0 - (V(U_0^+) - \{x,y\})$ by (3.4)(a). So we may assume $|E_0| \geq 2$. If we have case (ii), then $b \in H_2 \cap T_0 \neq \emptyset$, hence $||\widetilde{U}_0^*|| = 3|\widetilde{U}_0^*| - 7$ by (3.11)(a). For $\{t_1,t_2\} \subseteq {}'T_0$, there is a $b,\{t_1,t_2\}$-fan Q_1,Q_2 in $G_0(\widetilde{U}_0 \cup \{b,t_1,t_2\})$ ending in t_1,t_2 by (3.11)(b). For $z_0 \in \{t_1,t_2\}$, there is a $b,\{t_1,t_2\}$-fan Q_1,Q_2 in $G_0(\widetilde{U}_0 \cup \{b,t_1,t_2\})$ ending in t_1,t_2 by (3.16). Now P arises from P' by replacing $[b,t_i]$ with Q_i for $i=1,2$.

Now only case (iii) remains. If $[z_0,t] \in E(G_0)$, we have only to replace $[t_1,t_2]$ with a t_1,t_2-path in $\overline{\widetilde{U}}_0 - \{z_0,t\}$, which exists by (3.4)(a). So we may assume $[z_0,t] \notin E(G_0)$. Then $t \in H_2$ and by (3.11)(a) again $||\widetilde{U}_0^*|| = 3|\widetilde{U}_0^*| - 7$. Say, $b_2 \notin P'[z_0,b_1]$ and $p_1 = z_0$. Obviously, we may assume $t_1,t_2 \in P'[b_1,p_2]$; say, $t_2 \in P'[t_1,p_2]$. Since the graph $U' := \widetilde{U}_0^* - [z_0,t]$ is 2-connected by (3.11)(b), there are two disjoint $\{t,t_1\},\{z_0,t_2\}$-paths Q_1,Q_2 in U' by Menger's Theorem (cf. Theorem 3.3.1 in [1], for instance). Since $U' \subseteq G_0$ by (3.4)(b), it is easily checked that $P := (P' - \{[t,z_0],[t_1,t_2]\}) \cup Q_1 \cup Q_2$ is a p_1,p_2-path with the required property. ∎

Let P be a p_1,p_2-path as in (4.11). We may assume $p_1 \in P_1$ and $b_1 \in P[p_1,b_2]$. Since b_1,a_1,b_2,a_2 is a quadrangle in G, we get a $\dot{K}_4 \subseteq {}'G_0$ with $B(\dot{K}_4) = \{a_1,a_2,b_1,b_2\}$, using $P[b_1,b_2]$ and anyone of the paths P_1,P_2,P_3. Now we can easily extend such a $\dot{K}_4$ to a $\dot{K}_5 \subseteq G$. If $p_2 \in \mathring{P}_2 \cup \mathring{P}_3 \cup \{z_0\}$, we can enlarge for $p_2 \in \mathring{P}_2 \cup \mathring{P}_3$ the path $P[b_2,p_2]$ by $p_2 H_0 p_1$ and for $p_2 = z_0$ the b_2,p_2'-path $P[b_2,p_2)$ by $p_2' H_0 p_1$ to a b_2,p_1-path P_0, since $\{p_1,p_2\} \subseteq N_1$ and $\{p_1,p_2'\} \subseteq N_1$, respectively. But then we can extend a $\dot{K}_4$ as above by the $p_1,\{b_1,a_1,a_2,b_2\}$-fan $P[p_1,b_1], P_1[p_1,a_1], P_1[p_1,a_2], P_0$ to a $\dot{K}_5 \subseteq G$. This contradiction shows $p_2 \in \mathring{P}_1$, say, $p_2 \in P_1(p_1,a_2)$. But now we can expand a $\dot{K}_4$ as above by the $p_1,\{b_1,b_2,a_1,a_2\}$ - fan $P[p_1,b_1], P_1[p_1,p_2] \cup P[p_2,b_2], P_1[p_1,a_1], p_1 H_0 a_2$ to a $\dot{K}_5 \subseteq G$. This contradiction shows that case (i) cannot occur and we have $\triangle = 1$ by (4.8).

(ii) $\triangle = 1$.

Then $\kappa(G_1^*) \geq 3$ holds by (4.8) and (4.7)(c) and there is a $b \in B$ with $N(b) \supseteq A$ by definition of $\triangle$ and (4.10). Set $S_0 = \{s_1,s_2,s_3\}$ according to (4.3)(c). First we show

(4.12) *There are a b,S_0-fan Q_1,Q_2 in ${}'\overline{G}_1$ and an S_0-path Q_0 in ${}'\overline{G}_1$ with $\mathring{Q}_0 \cap (Q_1 \cup Q_2) = \emptyset$.*

Proof. First assume there is a circuit C in ${}'\overline{G}_1$ containing S_0. If $b \in C$, the existence of such paths Q_0,Q_1,Q_2 is obvious. If $b \notin C$, we can find a b,C-fan Q_1',Q_2' of order 2 in ${}'\overline{G}_1$, since ${}'\overline{G}_1 = {}'G_1^*$ is 2-connected. Since there is an S_0-path $Q_0 \subseteq C$ with

$\overset{\circ}{Q}_0 \cap (Q'_1 \cup Q'_2) = \emptyset$, the b, C-fan Q'_1, Q'_2 can be obviously enlarged to a b, S_0-fan Q_1, Q_2 which has only endvertices with Q_0 in common.

So we may assume that there is no circuit through S_0 in $'\overline{G}_1$. Since $\kappa('\overline{G}_1) \geq 2$, we can apply Theorem WM and choose C_1, C_2, C_3, C^1, C^2 with $s_i \in C_i$ for $i \in \mathbb{N}_3$ as in Corollary WM. We distinguish two cases corresponding to the different position of b, using the notation of Theorem WM and its Corollary.

(4.12.1) $$b \in C^1 \cup C^2.$$

Say $b \in C^1$. If $b = s^1_i$ for an $i \in \mathbb{N}_3$, say, $b = s^1_1$, then we can take, for instance, the b, S_0-fan $P^1_1, P^1_{1,2} \cup P^1_2$ and the S_0-path $P^2_2 \cup P^2_{2,3} \cup P^2_3$ in (4.12).

So we may assume $|C^1| > 1$, in particular, $|S^1| = 3$, and $b \neq s^1_i$ for all $i \in \mathbb{N}_3$. If there is a b, S^1-fan Q'_1, Q'_2 in C^1 ending in s^1_i, s^1_j, we can enlarge this fan by P^1_i, P^1_j to a b, S_0-fan ending in s_i, s_j. Then this fan and, for instance, the s_2, s_3-path $P^2_2 \cup P^2_{2,3} \cup P^2_3$ have the required property. So we may assume there is no b, S^1-fan of order 2 in C^1. This implies $|C^2| = 1$ by Corollary WM (iii), especially $|S^2| = 1$, and the existence of a $b, (S^1 \cup S^2)$-fan Q'_1, Q'_2 in $G(C^1 \cup S^2)$ ending in s^1_i, s^2, since $\kappa('\overline{G}_1) \geq 2$; say, $i = 1$. By Corollary WM (ii), there is a block B_0 of C^1 containing S^1. By our assumption, $b \notin B_0$. Hence there is a uniquely determined vertex $b' \in B_0 \cap Q'_1$ with $Q'_1[b, b') \cap B_0 = \emptyset$ and $Q'_1[b', s^1_1] \subseteq B_0$ (see figure 4.2) and $B_0 \cap Q'_2 = \emptyset$, since no b, B_0-path exists in $C^1 - b'$. If $b' = s^1_1$, then the b, S_0-fan $Q'_1 \cup P^1_1, Q'_2 \cup P^2_2$ and the S_0-path $P^1_2 \cup P^1_{2,3} \cup P^1_3$ have the property wanted, since $P^1_{2,3} \subseteq B_0 - s^1_1$. So we may assume $b' \neq s^1_1$. Then $|\{b'\} \cup S^1| = 4$ and by Menger's Theorem, (see, for instance, Theorem 3.3.1 in [1]) there are 2 disjoint $\{b', s^1_2\}, \{s^1_1, s^1_3\}$-paths Q, Q' in B_0; say, Q' is a b', s^1_1-path. Then the b, S_0-fan $Q'_1[b, b'] \cup Q' \cup P^1_1, Q'_2 \cup P^2_2$ and the S_0-path $P^1_2 \cup Q \cup P^1_3$ have the required property.

(4.12.2) $$b \in C_1 \cup C_2 \cup C_3.$$

We may assume, $b \in C_1$. If b and s_1 are in the same component of C_1, then we can find a $b, \{s_1, s^1_1, s^2_1\}$-fan Q_1, P^1 in $\overline{C}_1$ ending in s_1, s^i_1 by Theorem P; say,, $i = 1$. In this case we can take the b, S_0-fan $Q_1, P^1 \cup P^1_{1,2} \cup P^1_2$ and the S_0-path $P^2_2 \cup P^2_{2,3} \cup P^2_3$ in (4.12). Hence we may suppose that b and s_1 are in distinct components of C_1. But this cannot happen, because we could find a $\dot{K}_5 \subseteq G$ with $B(\dot{K}_5) = S_0 \cup \{s^1_1, s^2_1\}$ in the following way. Since $\kappa('\overline{G}_1) \geq 2$, there is a $b, \{s^1_1, s^2_1\}$-fan P^1, P^2 of order 2 in $'\overline{G}_1$ ending in s^1_1, s^2_1. Let $\mathfrak{F}^i$ be the $s^i_1, (S_0 \cup \{b\})$-fan $P^i, P^i_1, P^i_{1,2} \cup P^i_2, P^i_{1,3} \cup P^i_3$ of order 4 for $i = 1, 2$. Furthermore, there is a circuit C through S_0 in $G(\overline{F}_0 \cup H_0)$,

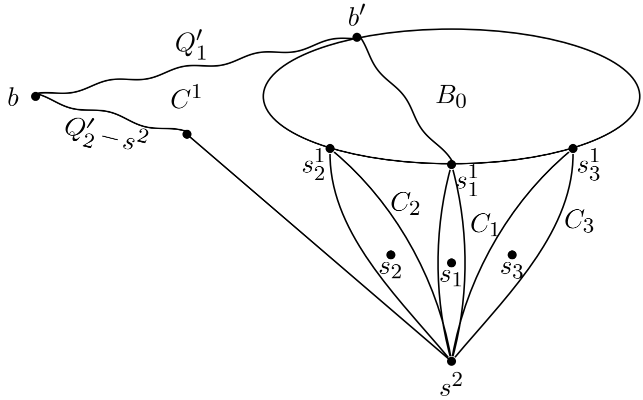

Fig. 4.2

since an $s_1,\{s_2,s_3\}$-fan of order 2 exists in $\overline{F}_0$ by (4.4) which can be joined to a circuit C by $s_2H_0s_3$. This circuit C together with $\mathfrak{F}^1\cup\mathfrak{F}^2$ forms a $\dot{K}_5$ as claimed. ∎

Now we can easily complete the proof of Theorem 2. Consider a b,S_0-fan Q_1,Q_2 and an S_0-path Q_0 as in (4.12); say, Q_0 is an s_1,s_2-path. Now we can apply (4.6) to the s_1,s_2-path Q_0 and $F:=\overline{F}_0^{Q_0}$. Since $\{s_1,s_2\}\cup A\subseteq V_{\geq 3}(F)$ by (4.3)(d), the existence of the b,F-fan $Q_1,Q_2,Q_3:=b,a_1,Q_4:=b,a_2$ for $A=\{a_1,a_2\}$ contradicts Proposition (4.6). ∎

5. Concluding remarks

One of the results of [4] we used to deduce Theorem 1 from Theorem 2 was that a minimum counterexample to Dirac's conjecture is 5-connected. This was known before (but not published) and prompted P. Seymour to state the following

(5.1) Conjecture (P. Seymour). *Every 5-connected, nonplanar graph contains a $\dot{K}_5$.*

Of course, this would also imply Theorem 1, but this conjecture remains open. It is true that every 4-connected, nonplanar graph is contractible to K_5 by a result of K. Wagner [14], but it does not necessarily contain a $\dot{K}_5$. The simplest example is the complete bipartite graph $K_{4,4}$. Since 5-connected graphs are difficult to handle, C. Thomassen tried to weaken this condition. Having (probably) $K_{4,4}$ in mind, he posed in [13] the following

Question [13]**.** *Let G be a 4-connected, nonplanar graph which contains no subdivision of K_5. Must G contain a set A of 4 vertices such that $G-A$ has 4 or more components?*

The following example gives a negative answer to this question.

(5.2) Example. Let C_n be a circuit of length $n\geq 3$ and let $G_n:=C_n[\overline{K}_2]$ arise from C_n replacing every vertex x with a set U_x of two non-adjacent vertices u_x^1 and u_x^2 and every edge $[x,y]\in E(C_n)$ with all edges $[u_x^i,u_y^j]$ for $i,j\in\mathbb{N}_2$. All these graphs G_n are 4-connected. They are nonplanar for $n\geq 4$ and for $n\geq 5, c(G_n-S)\leq 3$ for all $S\in\mathfrak{P}_4(V(G_n))$. (Note $G_4\cong K_{4,4}$.) G_5 contains a $\dot{K}_5$, but one can check that G_n does not for $n\geq 6$ (a proof is sketched below). So the graphs G_n for $n\geq 6$ answer the above question in the negative.

(Assume there is a $\dot{K}_5\subseteq G_n$ for an $n\geq 6$. Every separating set $S\in\mathfrak{P}_4(V(G_n))$ has the form $S=U_x\cup U_y$ for not adjacent vertices $x\neq y$ in C_n. There cannot be a separating set $S\in\mathfrak{P}_4(V(G_n))$ such that there is an S-fragment F of G_n with $|V(F)\cap B(\dot{K}_5)|\geq 2$ and $|V(\widetilde{F})\cap B(\dot{K}_5)|\geq 2$. Hence there is an $x\in C_n$ with $U_x\subseteq B(\dot{K}_5)$. Suppose there are even two vertices $x\neq y$ in C_n with $U_x\cup U_y\subseteq B(\dot{K}_5)$. Then $[x,y]\in E(C_n)$ and for the vertex $z\in B(\dot{K}_5)-(U_x\cup U_y)$, every $z,(U_x\cup U_y)$-fan of order 4 in G_n contains $N_{G_n}(U_x)$ or $N_{G_n}(U_y)$. This contradiction (since we need still a u_x^1,u_x^2-path and a u_y^1,u_y^2-path) shows that there is exactly one $x\in C_n$ with $U_x\subseteq B(\dot{K}_5)$. Now it is easy to exclude also this case, for instance, considering the separating set $N_{G_n}(U_x)$.)

The graphs G_n of example (5.2) have separating sets $S\in\mathfrak{P}_4(V(G_n))$ with $c(G_n-S)=3$. But one can also construct 4-connected, nonplanar graphs G not containing a $\dot{K}_5$, such that $c(G-S)\leq 2$ holds for all $S\in\mathfrak{P}_4(V(G))$.

(5.3) Example. For every integer $n\geq 6$, we define a graph $Z_n:=(\mathbb{Z}_n,\{[i,i+j]:i\in\mathbb{Z}_n$ and $j=1,2\in\mathbb{Z}_n\})$, where $\mathbb{Z}_n$ denotes the integers modulo n. Then Z_n is 4-regular, $\kappa(Z_n)=4$, and the sets $\{i,i+1,j,j+1\}\in\mathfrak{P}_4(\mathbb{Z}_n)$ with $1\leq i\leq n-3, i+3\leq j\leq n$, and $i\neq j+1,j+2\in\mathbb{Z}_n$ are the separating sets of Z_n with four vertices. So $c(Z_n-S)\leq 2$ for all $S\in\mathfrak{P}_4(\mathbb{Z}_n)$. For n odd, Z_n is nonplanar. One can prove that Z_n does not contain a $\dot{K}_5$ for all $n\geq 6$. Hence Z_7 is the simplest example of a 4-connected, nonplanar graph G with $c(G-S)\leq 2$ for all $S\in\mathfrak{P}_4(V(G))$, not containing a $\dot{K}_5$. (Note that Z_7 is even 4^+-connected.) But we can also construct graphs with these properties containing vertices of degree 5. Let the graphs H_n arise from Z_n by addition of a further vertex z and edges $[z,x]$ for all $x\in X\subseteq\mathbb{Z}_n$, where X has the property that $d_{Z_n}(x,y)\geq 3$ for all $\{x,y\}\in\mathfrak{P}_2(X), d_G(x,y)$ denoting the distance of x and y in G. Then H_n does not contain a $\dot{K}_5$. Such a nonplanar H_n is displayed for $n=20$ in figure 5.1.

For the proof that H_n does not contain a $\dot{K}_5$, one can proceed in the following way: Assume there is a $\dot{K}_5\subseteq H_n$. Then there is an $x\in B(\dot{K}_5)$ such that $\dot{K}_5-x$ contains a $\dot{K}_4\subseteq Z_n$. Now one can check successively that it is not possible that

(a) $\dot{K}_4$ contains two triangles with a common edge;

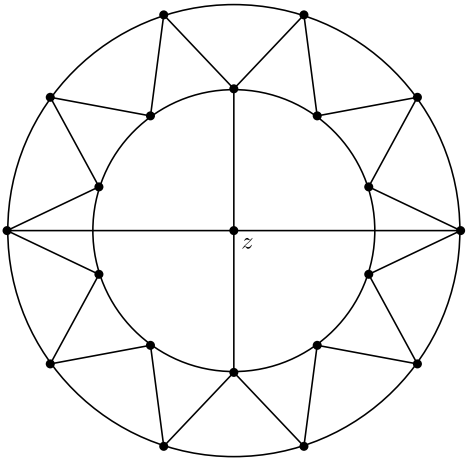

Fig. 5.1

(b) $\dot{K}_4$ contains a triangle;

(c) $\dot{K}_4$ has two branch vertices $i, i+1 \in \mathbb{Z}_n$.

On the other hand, there is an $i \in \mathbb{Z}_n$ so that $\{i, i+1\} \subseteq B(\dot{K}_4)$ holds, since otherwise a $j \in B(\dot{K}_4)$ would exist such that $j-1, j+1 \notin B(\dot{K}_4)$ are on the circuit in $\dot{K}_4 - j$.

It might be possible that Seymour's conjecture ressembles a property of 4-connected graphs without certain subgraphs, which have on the other side the property that the existence of these subgraphs in a (nonplanar) 5-connected graph gives enough information to deduce the existence of a $\dot{K}_5$. R. Thomas uttered in a conversation the idea that perhaps every 4-connected graph without triangles has a $\dot{K}_5$. This is not true as the graphs of example (5.2) (and also other graphs) show. But perhaps every graph G with $\kappa(G) \geq 4$ and $\tau(G) \geq 5$ contain a $\dot{K}_5$. In [6], I stated the stronger conjecture that every graph G with $\delta(G) \geq 4$ and $\tau(G) \geq 5$ contains a $\dot{K}_5$, and I proved in [7] that G with $\delta(G) \geq 4$ contains a $\dot{K}_5$, if $\tau(G)$ is large enough. Also the following could be true.

(5.4) Conjecture. *Every graph G with $\delta(G) \geq 5$, but without K_3 has a $\dot{K}_5$.*

An even stronger result might hold.

(5.5) Question. *Does every graph G with $\delta(G) \geq 5$, but without K_4^- contain a $\dot{K}_5$?*

There is a reason to consider (5.4) less probable than the conjecture mentioned in the paragraph before: in [6], I conjectured that every graph G with $||G|| \geq 2|G| - 3$ and $5 \leq \tau(G) < \infty$ contains a $\dot{K}_5$, but the condition $\tau(G) \geq 4$ does not essentially decrease the maximal edge number in a graph without $\dot{K}_5$, as $K_{3,n-3}$ shows. But perhaps one gets an essentially lower bound, if one also excludes these graphs as subgraphs.

(5.6) Question. *Does every graph G with $||G|| > \frac{12}{5}(|G|-2)$ and $|G| \geq 4$ contain a K_4^-, a $K_{2,3}$ or a $\dot{K}_5$?*

From Theorem 1 and the determination of the extremal graphs mentioned in the subsequent to Theorem 2, we recognize that the 5-connectedness in Theorem 2 is superfluous. Perhaps Theorem 2 can be improved in the following way.

(5.7) Question. *Does every 4-connected graph G with $||G|| > \frac{12}{5}(|G|-2)$ contain a K_4^-, a $K_{3,3}$ or a $\dot{K}_5$?*

I emphasize that $K_{2,3}$ in (5.6) cannot be replaced by $K_{3,3}$: one can specify infinitely many graphs G with $||G|| = \frac{8}{3}(|G|-3)$ containing neither a K_4^- nor a $K_{3,3}$ nor a $\dot{K}_5$, but these are not 4-connected. These questions cannot be answered in the negative by a planar graph, since it is easily checked that $||G|| \leq \frac{12}{5}(|G|-2)$ for every planar graph G with $|G| \geq 4$ not containing a K_4^-. The linear bound in (5.6) and (5.7) would be best possible, since one can construct infinitely many 4-connected, planar graphs G with $||G|| = \frac{12}{5}(|G|-2)$, but without K_4^- and without $K_{2,3}$. I should believe that a positive answer to anyone of the last three questions would offer the possibility to deduce (5.1).

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